# Introduction to quantum information processing

## Quantum Channels

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# Outline

# Last time...

- Mixed states are density operators $\rho$, including pure states as $|\psi\rangle\langle\psi|$.
- A generalized measurement (or POVM) is $\{E_j\}_{j=1}^{r}$ with $E_j \geq 0$ and

$$\sum_j E_j = \mathbb{1}.$$

- Probability is computed via Born's rule:

$$\Pr_\rho\{E_j\} = \operatorname{tr}(E_j \rho).$$

- The marginal (or partial trace) has $\operatorname{tr}(E \cdot \operatorname{tr}_B(\rho)) = \operatorname{tr}((E \otimes \mathbb{1}_B)\rho)$.
- Mixed state can be purified: $\rho = \operatorname{tr}_B(|\psi\rangle\langle\psi|)$ for some $|\psi\rangle \in \mathfrak{H}_A \otimes \mathfrak{H}_B$.

# Outline

# Unitary evolution

Dynamics of a pure state is given by unitary maps $U : \mathfrak{H} \to \mathfrak{H}$.

Writing a pure state as a mixed state $\rho = |\psi\rangle\langle\psi|$ unitary maps act as

$$|\psi\rangle\langle\psi| \mapsto (U|\psi\rangle)(\langle\psi|U^\dagger) = U(|\psi\rangle\langle\psi|)U^\dagger.$$

And so on general density matrices, we can define unitary action by

$$\rho \mapsto U\rho U^\dagger.$$

This is consistent with writing mixed states as ensembles.

- Note this is *not* an operator on the Hilbert space $\mathfrak{H}$,
- it is a linear map on the space of operators (or matrices) on $\mathfrak{H}$.
- Some call these *superoperators*, but we'll use the term "channel."

But are there more general forms of dynamics for mixed states?

# EIN ANDERE GEDANKENEXPERIMENT

Alice has a new idea. Now we give her a qubit and she flips two coins:

on HH she just gives us the qubit back,

on HT she applies $X$ and then returns the result,

on TH she applies $Y$ and then returns the result,

on TT she applies $Z$ and then returns the result.

However she does not tell us the result of the coins. What do we get?

If $\rho$ is the state we give Alice, then she returns the state:

$$\rho' = \underbrace{\tfrac{1}{4}\rho}_{\text{(HH)}} + \underbrace{\tfrac{1}{4}X\rho X}_{\text{(HT)}} + \underbrace{\tfrac{1}{4}Y\rho Y}_{\text{(TH)}} + \underbrace{\tfrac{1}{4}Z\rho Z}_{\text{(TT)}}.$$

Okay, what did this actually do to our state?

# Ein andere Gedankenexperiment

Let us write our initial qubit in Bloch sphere coordinates:

$$\rho = \frac{1}{2}\mathbb{1} + r_x X + r_y Y + r_z Z.$$

Then Alice returned:

$$\begin{aligned}\rho' &= \frac{1}{8}(\mathbb{1} + r_x X + r_y Y + r_z Z) \\ &+ \frac{1}{8}(XX + r_x XXX + r_y XYX + r_z XZX) \\ &+ \frac{1}{8}(YY + r_x YXY + r_y YYY + r_z YZY) \\ &+ \frac{1}{8}(ZZ + r_x ZXZ + r_y ZYZ + r_z ZZZ) = \frac{1}{2}\mathbb{1}.\end{aligned}$$

It doesn't matter what state we put in, we *always* get $\frac{1}{2}\mathbb{1}$ out!

# Outline

# Quantum channels

We use the idea "purifying" the dynamics:

- Given a mixed state $\rho$, we add an ancilla $\rho \otimes |\psi\rangle\langle\psi|$.
- The "pure" dynamics is unitary: $U(\rho \otimes |\psi\rangle\langle\psi|)U^\dagger$.
- Finally we trace out the ancillary space $\mathcal{C}(\rho) = \mathrm{tr}_B[U(\rho \otimes |\psi\rangle\langle\psi|)U^\dagger]$.

Let $\{|\chi_j\rangle\}$ be a basis of the ancillary space $\mathfrak{H}_B$.

- Define a linear map as follows: for $|\phi\rangle \in \mathfrak{H}_A$.

$$A_j|\phi\rangle = \langle\chi_j|U(|\phi\rangle \otimes |\psi\rangle) \in \mathfrak{H}_A.$$

- Take the spectral decomposition $\rho = \sum_\mu \lambda_\mu |\phi_\mu\rangle\langle\phi_\mu|$.
- Compute:

$$\begin{aligned}\mathcal{C}(\rho) = \mathrm{tr}_B[U(\rho \otimes |\psi\rangle\langle\psi|)U^\dagger] &= \sum_{j\mu} \lambda_\mu \langle\chi_j|U(|\phi_\mu\rangle\langle\phi_\mu| \otimes |\psi\rangle\langle\psi|)U^\dagger|\chi_j\rangle \\ &= \sum_{j\mu} \lambda_\mu A_j|\phi_\mu\rangle\langle\phi_\mu|A_j^\dagger = \sum_j A_j \rho A_j^\dagger.\end{aligned}$$

# QUANTUM CHANNELS

A *quantum channel* is the analogue of a unitary map for mixed states.

- The channel is defined by the "superoperator" $\mathcal{C}(\rho) = \sum_j A_j \rho A_j^\dagger$;
- this is always positive, but we need $\sum_j A_j^\dagger A_j = \mathbb{1}$ to preserve traces.
- The $A_j$ are called *Kraus*, or *Kraus-Choi*, operators.

*Note*: quantum channels can map between *different* Hilbert spaces:

- if $A_j : \mathfrak{H} \to \mathfrak{K}$ then $A_j^\dagger : \mathfrak{K} \to \mathfrak{H}$; then $A_j \rho A_j^\dagger$ is an operator on $\mathfrak{K}$.
- So $\mathcal{C}$ maps densities on $\mathfrak{H}$ to densities on $\mathfrak{K}$.

Note that if $\dim \mathfrak{H} \neq \dim \mathfrak{K}$ then "purifying" the dynamics doesn't work:

- we can't have unitaries between different dimensional spaces,
- however we can still find Kraus operators by a different means.

This is called Stinespring dilation (it's in the appendix to these notes).

# Unitary freedom of Kraus matrices

A channel is defined by the formula $\mathcal{C}(\rho) = \sum_j A_j \rho A_j^\dagger$.

- The Kraus operators $A_j$ are *not* unique!
- Recall the motivation: if $\{|\chi_j\rangle\}$ is an orthonormal basis

$$A_j|\phi\rangle = \langle\chi_j|U(|\phi\rangle \otimes |\psi\rangle).$$

- So the Kraus operators really depend on this basis.

Let $(u_{jk})$ be the components of an arbitrary unitary matrix.

- Define $\hat{A}_j = \sum_k u_{jk} A_k$. Then we have $A_j = \sum_k u_{kj}^* \hat{A}_k$.
- We compute:

$$\mathcal{C}(\rho) = \sum_j A_j \rho A_j^\dagger = \sum_{jk\ell} u_{kj}^* u_{\ell j} \hat{A}_k \rho \hat{A}_\ell = \sum_k \hat{A}_k \rho \hat{A}_k$$

(*Note*: for this computation, we really only needed $\sum_j u_{kj}^* u_{\ell j} = \delta_{kl}$.)

# Process matrices

When building devices, we try to explain what happens using channels.

Consider a quantum channel $\mathcal{C}$ on densities on $\mathfrak{H}$.

- Let $L(\mathfrak{H})$ be all linear maps on the Hilbert space $\mathfrak{H}$.
- It has dimension $n^2$ where $n = \dim \mathfrak{H}$.
- Let $\{E_\alpha\}_{\alpha=1}^{n^2}$ be a basis for $L(\mathfrak{H})$.
- Expand any Kraus operator in term of this basis $A_j = \sum_\alpha c_{j\alpha} E_\alpha$.

Then every quantum channel also has the form

$$\mathcal{C}(\rho) = \sum_j A_j \rho A_j^\dagger = \sum_{\alpha\beta} \chi_{\alpha\beta} E_\alpha \rho E_\beta^\dagger$$

where $\chi_{\alpha\beta} = \sum_j c_{j\alpha} c_{j\beta}^*$ is the *process matrix* of the channel.

One often sees empirically estimates the process matrix to describe the dynamics of a quantum system. This is called "process tomography."

# Outline

# State preparation as a channel

Suppose we want to prepare a system in state $|\psi\rangle \in \mathfrak{H}$.

- Let $|0\rangle$ be the basis for the trivial Hilbert space $\mathbb{C}$.
- Define (one) Kraus operator $A = |\psi\rangle\langle 0|$ for a quantum channel $\mathcal{P}$.
- Then this channel is just state preparation:

$$\mathcal{P}(|0\rangle\langle 0|) = A|0\rangle\langle 0|A^\dagger = |\psi\rangle\langle\psi|.$$

We represent *classical* information as a diagonal mixed state.

- This is a basis dependent notion, e.g. for orthonormal basis $\{|\phi_j\rangle\}_{j=1}^n$
- a classical distribution $p$ on $\{1,\ldots,n\}$ becomes $\rho = \sum_j p(j)|\phi_j\rangle\langle\phi_j|$.

We can prepare any ensemble of states $\{|\psi_j\rangle\}$ using a quantum channel:

- Use the Hilbert space $\mathbb{C}^n$ with "classical" basis $\{|\phi_j\rangle\}_{j=1}^n$,
- form the Kraus operators $A_j = |\psi_j\rangle\langle\phi_j|$.

# Measurement redux

State preparation is a quantum channel with classical domain.

- So what about a quantum channel with classical range.
- I.e. the channel converts quantum states into classical probabilities.
- We claim this is great way to view measurements.

Let $\mathfrak{H}$ be a Hilbert space and $\{|\phi_j\rangle\}$ our measurement basis.

- Define a channel $\mathcal{M}$ using Kraus operators $\Pi_j = |\phi_j\rangle\langle\phi_j|$.
- Then we compute:

$$\mathcal{M}(\rho) = \sum_j \Pi_j \rho \Pi_j = \sum_j |\phi_j\rangle\langle\phi_j|\rho|\phi_j\rangle\langle\phi_j| = \sum_j p(j)|\phi_j\rangle\langle\phi_j|$$

where $p(j) = \langle\phi_j|\rho|\phi_j\rangle = \mathrm{tr}(\Pi_j\rho)$.

The channel $\mathcal{M}$ encodes what will happen if a measurement is performed.

# POVMs as quantum channels

This works for *any* set of vectors $\{|\phi_j\rangle\}_{j=1}^r$ with $\sum_{j=1}^r |\phi_j\rangle\langle\phi_j| = \mathbb{1}$.

- Write $\{|j\rangle\}_{j=1}^r$ be an orthonormal basis of $\mathbb{C}^r$.
- Define $A_j = |j\rangle\langle\phi_j|$. Then $\sum_{j=1}^r A_j^\dagger A_j = \sum_{j=1}^r |\phi_j\rangle\langle j|j\rangle\langle\phi_j| = \mathbb{1}$.
- Therefore $\mathcal{G}(\rho) = \sum_{j=1}^r A_j \rho A_j^\dagger$ is a quantum channel.

But as with our previous computation:

$$\mathcal{G}(\rho) = \sum_{j=1}^{r} |j\rangle\langle\phi_j|\rho|\phi_j\rangle\langle j| = \sum_{j=1}^{r} \operatorname{tr}(|\phi_j\rangle\langle\phi_j|\rho) \cdot |j\rangle\langle j|.$$

So $E_j = |\phi_j\rangle\langle\phi_j|$ is a POVM, and $\mathcal{G}$ is its "generalized" measurement.

But what happens when our measurements are not just projectors?

- Then some quantum information remains after the measurement.
- In other words, the channel is a *partial* measurement.

# Partial measurements and quantum channels

## Lemma

*Let $E \geq 0$ on a Hilbert space $\mathfrak{H}$. Then $E = A^\dagger A$ for some operator $A$ on $\mathfrak{H}$.*

Let $\{E_j\}_{j=1}^r$ be a POVM, and write $E_j = A_j^\dagger A_j$.

- Born's rule has $p(j) = \operatorname{tr}(E_j\rho) = \operatorname{tr}(A_j^\dagger A_j \rho) = \operatorname{tr}(A_j \rho A_j^\dagger)$.
- As before let $\{|j\rangle\}_{j=1}^r$ is the "classical" basis of $\mathbb{C}^r$.
- Define $\bar{A}_j : \mathfrak{H} \to \mathfrak{H} \otimes \mathbb{C}^r$ by $\bar{A}_j|\psi\rangle = A_j|\psi\rangle \otimes |j\rangle$.
- Define the channel

$$\mathcal{G}(\rho) = \sum_{j=1}^{r} \bar{A}_j \rho \bar{A}_j^\dagger = \sum_{j=1}^{r} A_j \rho A_j^\dagger \otimes |j\rangle\langle j| = \sum_{j=1}^{r} \frac{A_j \rho A_j^\dagger}{\operatorname{tr}(A_j \rho A_j^\dagger)} \otimes p(j)|j\rangle\langle j|.$$

The classical factor has our probability of observing a particular outcome.

The quantum factor has the state after conditioning seeing that outcome.

# The Stinespring Dilation theorem

We defined a quantum channel by purifying. But there is an axiomatic way:

- (Convexity) $\mathcal{C}(\sum_j p_j\rho_j) = \sum_j \mathcal{C}(\rho_j)$ whenever $p_j > 0$ with $\sum_j p_j = 1$.
- (Trace-preserving) $\mathrm{tr}(\mathcal{C}(\rho)) = \mathrm{tr}(\rho) = 1$.
- (Completely positive) For all $m > 0$, we have $(\mathcal{C} \otimes \mathbb{1}_{\mathbb{C}^m})(\rho) \geq 0$ whenever $\rho \geq 0$ on $\mathfrak{H} \otimes \mathbb{C}^m$.

## Theorem (Stinespring)

*Let $\mathcal{C}$ map densities on $\mathfrak{H}$ to densities on $\mathfrak{K}$ be convex, trace-preserving, and completely positive. Then there exists a linear map $V : \mathfrak{H} \to \mathfrak{K} \otimes \mathbb{C}^r$ so that $\mathcal{C}(\rho) = \mathrm{tr}_{\mathbb{C}^r}(V\rho V^\dagger)$.*

In general $V$ is not unitary (dimensions don't match), but this form is enough to compute Kraus operators and so get a quantum channel.

Note that $V$, and even $r$, is not unique. However different Stinespring dilations are equivalent under "partial isometry" (the analogue of unitary between different dimensional spaces).

# Next time...

- Noise channels and process tomography.
- Fidelity.
- Trace distance.