Introduction to quantum information processing

Measurements and quantum probability

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Outline

Probability and measurement

As taught in quantum mechanics

In traditional quantum mechanics courses:

1. A system is associated with a Hilbert space $\mathfrak{H}$.
   - In quantum information we typically work with finite dimensional spaces.
2. Probability is determined by the "state" of the system $|\psi\rangle \in \mathfrak{H}$.
   - This is a unit vector, but only determined up to "phase."
3. Observables are given by Hermitian operators A, and $\langle A\rangle = \langle\psi|A|\psi\rangle$.
   - In particular, if $|\phi\rangle$ is an eigenvector of A with eigenvalue λ then

$$\Pr\{\text{Getting outcome } \lambda \text{ upon measuring } A\} =$$
$$\Pr\{\text{Seeing the state } |\phi\rangle \text{ upon observing } A\} = |\langle\psi|\phi\rangle|^2.$$

   - This produces the idea of a "measurement" basis: the eigenbasis of A.

Outline

1. Probability
2. Density Operators
3. Review: the Spectral Theorem

Ein Gedankenexperiment

Alice flips a (fair) coin, and without telling us the result does the following:

- on heads, she prepares the state $|0\rangle$; and
- on tails, she prepares the state $|1\rangle$.

Then sends the resulting state. We make a measurement. What do we get?

Say our measurement basis is

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle,$$
$$|\phi^{\perp}\rangle = \beta^{*}|0\rangle - \alpha^{*}|1\rangle.$$

Then

$$\Pr\{\text{Seeing } |\phi\rangle\} = \underbrace{\tfrac{1}{2}|\langle 0|\phi\rangle|^2}_{\text{Heads}} + \underbrace{\tfrac{1}{2}|\langle 1|\phi\rangle|^2}_{\text{Tails}} = \frac{|\alpha|^2+|\beta|^2}{2} = \frac{1}{2}$$

$$\Pr\{\text{Seeing } |\phi^{\perp}\rangle\} = \underbrace{\tfrac{1}{2}|\langle 0|\phi^{\perp}\rangle|^2}_{\text{Heads}} + \underbrace{\tfrac{1}{2}|\langle 1|\phi^{\perp}\rangle|^2}_{\text{Tails}} = \frac{|\beta|^2+|\alpha|^2}{2} = \frac{1}{2}.$$

Quantum probability

Hmmm... Alice managed to prepare a state where *any* measurement basis $\{|\phi\rangle, |\phi^\perp\rangle\}$ we use gives us the statistics:

$$\Pr\{\text{Seeing } |\phi\rangle\} = \Pr\{\text{Seeing } |\phi^\perp\rangle\} = \frac{1}{2}.$$

So what is this state?

Suppose this state is $|\psi\rangle$. But then measuring in the basis $\{|\psi\rangle, |\psi^\perp\rangle\}$ gives

$$\Pr\{\text{Seeing } |\psi\rangle\} = 1 \text{ and } \Pr\{\text{Seeing } |\psi^\perp\rangle\} = 0,$$

not 50-50 as we've already shown it must be.

So this state is never of the form $|\psi\rangle$! What is it?

Answer: states of the form $|\psi\rangle$ are special, called "pure" states. There's many more quantum states than just these. Alice prepared something called a "mixed" state.

MIXED STATES VERSUS PURE STATES

Let figure it out what it is by pushing symbols around:

$$\begin{aligned}
\Pr\{\text{Seeing } |\phi\rangle\} &= \frac{1}{2}|\langle 0|\phi\rangle|^2 + \frac{1}{2}|\langle 1|\phi\rangle|^2 \\
&= \frac{1}{2}(\langle 0|\phi\rangle)^* \langle 0|\phi\rangle + \frac{1}{2}(\langle 1|\phi\rangle)^* \langle 1|\phi\rangle \\
&= \frac{1}{2}\langle \phi|0\rangle\langle 0|\phi\rangle + \frac{1}{2}\langle \phi|1\rangle\langle 1|\phi\rangle \\
&= \langle\phi| \left(\frac{1}{2}|0\rangle\langle 0|\right)|\phi\rangle + \langle\phi| \left(\frac{1}{2}|1\rangle\langle 1|\right)|\phi\rangle \\
&= \langle\phi| \left(\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)\right)|\phi\rangle \\
&= \langle \phi|\rho|\phi\rangle
\end{aligned}$$

where

$$\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|).$$

Mixed states versus pure states

What's with this notation? Let's really compute this mixed state:

$$\frac{1}{2}\langle\phi|0\rangle\langle 0|\phi\rangle = \frac{1}{2}\left[\begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right]$$
$$= \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}\left[\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\right]\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

So $\frac{1}{2}|0\rangle\langle 0|$ is the matrix (or operator)

$$\frac{1}{2}|0\rangle\langle 0| = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$$

And similarly for $\frac{1}{2}|1\rangle\langle 1|$, and so

$$\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\cdot \mathbb{1}.$$

MIXED STATES VERSUS PURE STATES

Now things are falling into place.

- The state Alice prepared is an operator $\rho = \frac{1}{2} \cdot \mathbb{1}$.
- We've shown the general probability rule

$$\Pr\{\text{Seeing } |\phi\rangle\} = \langle\phi|\rho|\phi\rangle.$$

- So we simply compute: for any $|\phi\rangle$,

$$\frac{1}{2}\langle\phi|\mathbb{1}|\phi\rangle = \frac{1}{2}\langle\phi|\phi\rangle = \frac{1}{2}.$$

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Mixed states and ensembles

Instead, imagine Alice prepared a huge number of quantum states:

- half of them are prepared in $|0\rangle$, and the other half in $|1\rangle$;
- we take one at random and measure it.

In general: an *ensemble* is a collection $\{(|\psi_j\rangle, p_j)\}_{j=1}^N$

- $|\psi_j\rangle \in \mathfrak{H}$ is any pure state, and
- p_j is the proportion of the ensemble prepared in $|\psi_j\rangle$.

Then our statistics are given by

$$\begin{aligned}\Pr\{\text{Seeing } |\phi\rangle\} &= \sum_{j=1}^N p_j |\langle\psi_j|\phi\rangle|^2 \\ &= \sum_{j=1}^N p_j \langle\phi|\psi_j\rangle\langle\psi_j|\phi\rangle = \langle\phi| \left(\sum_{j=1}^N p_j |\psi_j\rangle\langle\psi_j|\right) |\phi\rangle.\end{aligned}$$

So ensembles are (generally) types of mixed states.

Positive operators

The state of an ensemble has the form $\rho = \sum_{j=1}^{N} p_j |\psi_j\rangle\langle\psi_j|$.

- This operator is Hermitian since p_j is real.

Definition

A Hermitian operator ρ is *positive* (denoted $\rho \geq 0$) if all its eigenvalues are nonnegative.

Why isn't this called nonnegative? I don't know.

Proposition

An operator $\rho \geq 0$ if and only if for every $|\phi\rangle$ we have $\langle\phi|\rho|\phi\rangle \geq 0$.

We'll prove this soon, but we can apply it now.

- The state of any ensemble is positive. In fact,

$$\langle\phi|\rho|\phi\rangle = \sum_{j=1}^{N} p_j |\langle\psi_j|\phi\rangle|^2 = \Pr\{\text{Seeing } |\phi\rangle\} \geq 0.$$

Trace

Recall from the linear algebra:

- the trace of a matrix is the sum of its diagonal entries.
- Important fact 1: $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
- Important fact 2: $\mathrm{tr}(c_1 A + c_2 B) = c_1 \mathrm{tr}(A) + c_2 \mathrm{tr}(B)$.

Fact 1 implies that the trace can be computed in any basis:

- if P is the change-of-basis matrix, then

$$\mathrm{tr}(P^{-1}AP) = \mathrm{tr}((P^{-1}A)P) = \mathrm{tr}(P(P^{-1}A)) = \mathrm{tr}(PP^{-1}A) = \mathrm{tr}(A).$$

In Dirac notation the trace is expressed as follows.

- Let $\{|\phi_j\rangle\}$ be *any* basis of $\mathfrak{H}$ (typically we use an orthonormal one).
- The trace of an operator is given by

$$\mathrm{tr}(\rho) = \sum_j \langle \phi_j | \rho | \phi_j \rangle.$$

(*Warning*: in general $\langle \phi_j |$ refers to the "dual" basis; this can be a tricky to compute if the basis is not orthonormal!)

Density operators

The state of an ensemble is $\rho = \sum_{j=1}^{N} p_j |\psi_j\rangle\langle\psi_j|$.

Proposition

The state of any ensemble has trace one.

Now we use an *orthonormal* basis, $\{|\phi_k\rangle\}$. We first compute

$$\begin{aligned}\mathrm{tr}(|\psi_j\rangle\langle\psi_j|) &= \sum_k \langle\phi_k|\psi_j\rangle\langle\psi_j|\phi_k\rangle = \sum_k |\langle\psi_j|\phi_k\rangle|^2 \\ &= \|\phi_j\|^2 = 1 \text{ (Pythagoras' Theorem).}\end{aligned}$$

But then

$$\mathrm{tr}(\rho) = \mathrm{tr}\left(\sum_{j=1}^{N} p_j|\psi_j\rangle\langle\psi_j|\right) = \sum_{j=1}^{N} p_j \mathrm{tr}(|\psi_j\rangle\langle\psi_j|) = \sum_{j=1}^{N} p_j = 1.$$

Definition

A *density operator* is an operator ρ on $\mathfrak{H}$ with (i) $\rho \geq 0$ and (ii) $\mathrm{tr}(\rho) = 1$.

Example: densities on the Bloch sphere

Recall the Bloch sphere:

- a qubit can be written as $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle$,
- on the Bloch sphere it is $(r_x, r_y, r_z) = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$.

Consider $|\psi\rangle$ as an ensemble by itself $(|\psi\rangle, 1)$, so $\rho = |\psi\rangle\langle\psi|$. We compute:

$$\begin{aligned}
\rho &= \begin{pmatrix} \cos(\theta/2) \\ e^{i\varphi}\sin(\theta/2) \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & e^{-i\varphi}\sin(\theta/2) \end{pmatrix} \\
&= \begin{pmatrix} \cos^2(\theta/2) & e^{-i\varphi}\sin(\theta/2)\cos(\theta/2) \\ e^{i\varphi}\sin(\theta/2)\cos(\theta/2) & \sin^2(\theta/2) \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1+\cos\theta & \cos\varphi\sin\theta - i\sin\varphi\sin\theta \\ \cos\varphi\sin\theta + i\sin\varphi\sin\theta & 1-\cos\theta \end{pmatrix} \\
&= \frac{1}{2}(\mathbb{1} + \cos\theta Z + \cos\varphi\sin\theta X + \sin\varphi\sin\theta Y) \\
&= \frac{1}{2}(\mathbb{1} + r_x X + r_y Y + r_z Z).
\end{aligned}$$

So the Bloch sphere coordinates of $|\psi\rangle$ are just the coefficients of $\rho = |\psi\rangle\langle\psi|$ with respect to the Pauli matrices.

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Eigenvectors and eigenvalues

We're familiar with eigenvectors and eigenvalues of matrix $M\vec{v} = \lambda\vec{v}$.

- Dirac notation: $M|\psi\rangle = \lambda|\psi\rangle$.

Important fact: let H be Hermitian on $\mathfrak{H}$,

1. then $\mathfrak{H}$ has a orthonormal basis of eigenvectors for H, $\{|\psi_j\rangle\}_{j=1}^{\dim\mathfrak{H}}$,
2. and all the eigenvalues of H are real.

Proposition

The operator $\Pi = |\psi\rangle\langle\psi|$ *is the orthogonal projection onto* $\mathrm{span}\{|\psi\rangle\}$.

In other words: (i) $\Pi|\psi\rangle = |\psi\rangle$ and (ii) if $\langle\psi|\phi\rangle = 0$ then $\Pi|\phi\rangle = 0$.

Using the notation above $\Pi_j = |\psi_j\rangle\langle\psi_j|$ is an *eigenprojector* of H.

- The eigenvalue equation reads $H\Pi_j = \lambda_j\Pi_j$.
- Orthogonality implies $\Pi_j\Pi_k = 0$ whenever $j \neq k$.

Spectral measures

What if H has degenerate eigenvalues? E.g. $H = Z \otimes \mathbb{1}$:

- eigenvalue +1 has eigenvectors $|00\rangle$ and $|01\rangle$,
- eigenvalue -1 has eigenvectors $|10\rangle$ and $|11\rangle$.

This would come up when we "measure the first qubit of a register."

- The eigenprojection Π_{+1} maps onto span$\{|00\rangle, |01\rangle\}$.
- The eigenprojection Π_{-1} maps onto span$\{|10\rangle, |11\rangle\}$.

If $\{|\psi_k\rangle\}_{k=1}^{d}$ is a *orthonormal* basis of eigenvectors for eigenvalue λ:

- the eigenprojection is $\Pi_\lambda = |\psi_1\rangle\langle\psi_1| + \cdots + |\psi_d\rangle\langle\psi_d|$.
- Still, if $\lambda \neq \mu$ are eigenvalues then $\Pi_\lambda \Pi_\mu = 0$.

Definition

A *spectral (or projection-valued) measure* is a collection of orthogonal projections $\{\Pi_j\}$ such that (i) $\Pi_j \Pi_k = 0$ whenever $j \neq k$, and (ii) $\sum_j \Pi_j = \mathbb{1}$.

THE SPECTRAL THEOREM

Theorem (The Spectral Theorem)

Let H be Hermitian, $\sigma(H) = \{\lambda_1, \ldots, \lambda_r\}$ be its eigenvalues, and $\{\Pi_1, \ldots, \Pi_r\}$ be the associated eigenprojections. Then for any function f defined on $\sigma(H)$, we have $f(H) = \sum_{k=1}^{r} f(\lambda_k)\Pi_k$.

This is not just notation: consider $H^2 = \sum_{k=1}^{r} \lambda_k^2 \Pi_k$.

- I know how to calculate each side independently.
- The spectral theorem shows they always give the same answer.

More important examples:

- (spectral resolution) for $f(x) = x$ we have $H = \sum_{k=1}^{r} \lambda_k \Pi_k$;
- (square-root) if $\rho \geq 0$ and $f(x) = \sqrt{x}$ we have $\rho^{1/2} = \sum_{k=1}^{r} \sqrt{\lambda_k}\Pi_k$;
- (modulus) for $f(x) = |x|$ we have $|A| = \sum_{k=1}^{r} |\lambda_k|\Pi_k$;
- (propagation) for $f(x) = e^{ixt}$ we have $e^{iHt} = \sum_{k=1}^{r} e^{i\lambda_k t}\Pi_k$;
- (resolution-of-the-identity) for $f(x) = 1$ we have $\mathbb{1} = \sum_{k=1}^{r} \Pi_k$.

EXAMPLE: DENSITIES VERSUS ENSEMBLES

We claim every density operator is just an ensemble.

- Use the spectral resolution to write $\rho = \sum_{j=1}^{\dim \mathfrak{H}} \lambda_j |\psi_j\rangle\langle\psi_j|$.
- Here, if an eigenvalue is degenerate (say λ_k) we split up its projection $\Pi_k = |\psi_{k_1}\rangle\langle\psi_{k_1}| + \cdots + |\psi_{k_d}\rangle\langle\psi_{k_d}|$.
- Always $\lambda_j \geq 0$ since $\rho \geq 0$.

Trace can be computed in any basis, so use the basis $\{|\psi_j\rangle\}$:

$$\mathrm{tr}(\rho) = \sum_{jk} \lambda_j |\langle\psi_j|\psi_k\rangle|^2 = \sum_j \lambda_j = 1.$$

Therefore $\{(|\psi_j\rangle, \lambda_j)\}$ is an ensemble.

So we see (i) mixed states, (ii) ensembles, and (iii) density operators are all the same.

Positive operators

Some unfinished business:

Proposition

An operator $A \geq 0$ if and only if for every $|\psi\rangle$, we have $\langle\psi|A|\psi\rangle \geq 0$.

Proof. Suppose $\langle\psi|A|\psi\rangle \geq 0$ for all $|\psi\rangle$. Let λ be an eigenvalue of A, so that $A|\phi\rangle = \lambda|\phi\rangle$ for some $|\phi\rangle$. Then

$$0 \leq \langle\phi|A|\phi\rangle = \langle\phi|\lambda|\phi\rangle = \lambda.$$

Since λ was arbitrary, all eigenvalues of A are nonnegative.

Now suppose $A \geq 0$, and write $A = \sum_{j=1}^{n} \lambda_j |\phi_j\rangle\langle\phi_j|$ using the spectral theorem. Then since each $\lambda_j \geq 0$, every $|\psi\rangle$ has

$$\langle\psi|A|\psi\rangle = \sum_{j=1}^{n} \lambda_j |\langle\psi|\phi\rangle_j|^2 \geq 0. \quad \square$$

NEXT TIME...

- Schmidt decomposition.
- Partial trace.
- State purification.
- Superoperators.