Practice Problems – Type Systems
Sample Solution

1. (a) $\dfrac{}{\cdot \vdash 42 : \mathit{int}}$

   (b) $\dfrac{x : \mathit{int}, y : \mathit{int} \vdash y : \mathit{int}}{y : \mathit{int} \vdash \lambda x{:}\,\mathit{int}.y : \mathit{int} \to \mathit{int}}$

   (c) $\dfrac{\dfrac{\dfrac{}{y : \mathit{int}, x : \mathit{int} \vdash x : \mathit{int}}}{x : \mathit{int} \vdash \lambda y{:}\,\mathit{int}.x : \mathit{int} \to \mathit{int}}}{\cdot \vdash \lambda x{:}\,\mathit{int}.\lambda y{:}\,\mathit{int}.x : \mathit{int} \to \mathit{int} \to \mathit{int}}$

   (d) Let $A = + : \mathit{int} \to \mathit{int} \to \mathit{int}$. See Figure 1.

2. There are many possible examples, e.g., $(\lambda x{:}\,\mathit{int} \to \mathit{int}.42)\ (\lambda y{:}\,\mathit{int}.1\ 2)$.

3. (a) (as shown in problem set)

   (b)
   $$\dfrac{\dfrac{y : \beta, x : \alpha \vdash x : \alpha}{x : \alpha \vdash \lambda y.x : \beta \to \alpha}}{\cdot \vdash \lambda x.\lambda y.x : \alpha \to \beta \to \alpha}$$
   Solution: α and β are unconstrained
   Type : $\alpha \to \beta \to \alpha$.

   (c)
   $$\dfrac{\dfrac{\dfrac{y : \beta, x : \alpha \vdash x : \alpha \qquad y : \beta, x : \alpha \vdash y : \beta \qquad \alpha = (\beta \to \gamma)}{y : \beta, x : \alpha \vdash x\ y : \gamma}}{x : \alpha \vdash \lambda y.x\ y : \beta \to \gamma}}{\cdot \vdash \lambda x.\lambda y.x\ y : \alpha \to \beta \to \gamma}$$
   Solution: $\alpha = (\beta \to \gamma)$, and β and γ are unconstrained.
   Type: $(\beta \to \gamma) \to \beta \to \gamma$.

   (d)
   $$\dfrac{\dfrac{\dfrac{\dfrac{y : \beta, x : \alpha \vdash x : \alpha}{x : \alpha \vdash \lambda y.x : \beta \to \alpha}}{\cdot \vdash (\lambda x.\lambda y.x) : \alpha \to \beta \to \alpha} \qquad \cdot \vdash 3 : \mathit{int} \qquad (\alpha \to \beta \to \alpha) = (\mathit{int} \to \gamma)}{\cdot \vdash (\lambda x.\lambda y.x)\ 3 : \gamma} \qquad \cdot \vdash 42 : \mathit{int} \qquad \gamma = (\mathit{int} \to \delta)}{\cdot \vdash (\lambda x.\lambda y.x)\ 3\ 42 : \delta}$$
   Solution: $\alpha = \beta = \delta = \mathit{int}$, $\gamma = \mathit{int} \to \mathit{int}$.
   Type: int.

$$
\dfrac{
  \dfrac{
    \dfrac{}{f : int \to int, A \vdash f : int \to int}
    \quad
    \dfrac{}{f : int \to int, A \vdash 42 : int}
  }{f : int \to int, A \vdash f\ 42 : int}
}{A \vdash (\lambda f.f\ 42) : (int \to int) \to int}
\quad
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{x : int, A \vdash + : int \to int \to int}
      \quad
      \dfrac{}{x : int, A \vdash x : int}
    }{x : int, A \vdash +\ x : int \to int}
    \quad
    \dfrac{}{x : int, A \vdash 3 : int}
  }{x : int, A \vdash +\ x\ 3 : int}
}{A \vdash (\lambda x.+\ \ x\ 3) : int \to int}
\over
A \vdash (\lambda f.f\ 42)\ (\lambda x.+\ \ x\ 3) : int
$$

Figure 1: Derivation for 2.a.iv. The types of function arguments have been omitted for space reasons.