

Assignment 2

Please submit it electronically to ELMS. This assignment is 9% in your total points. For the simplicity of the grading, the total points for the assignment is 90.

Problem 1. *The Hadamard gate and qubit rotations*

1. (3 points) Suppose that $(n_x, n_y, n_z) \in \mathcal{R}^3$ is a unit vector and $\theta \in \mathcal{R}$. Show that

$$e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)} = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) (n_x X + n_y Y + n_z Z).$$

2. (5 points) Find a unit vector $(n_x, n_y, n_z) \in \mathcal{R}^3$ and numbers $\phi, \theta \in \mathcal{R}$ so that

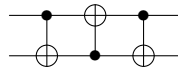
$$H = e^{i\phi} e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)},$$

where H denotes the Hadamard gate. What does this mean in terms of the Bloch sphere?

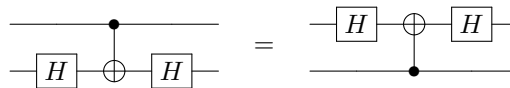
3. (7 points) Write the Hadamard gate as a product of rotations about the x and y axes. In particular, find $\alpha, \beta, \gamma, \phi \in \mathcal{R}$ such that $H = e^{i\phi} R_y(\gamma) R_x(\beta) R_y(\alpha)$.

Problem 2. *Circuit identities.*

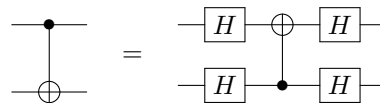
1. (5 points) Show that the following circuit swaps two qubits:



2. (5 points) Verify the following circuit identity:



3. (5 points) Verify the following circuit identity:



Give an interpretation of this identity.

Problem 3. *Universality of gate sets.* Prove that each of the following gate sets either is or is not universal. You may use the fact that the set $\{\text{CNOT}, H, T\}$ is universal.

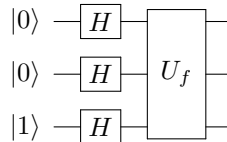
1. (5 points) $\{H, T\}$
 2. (5 points) $\{\text{CNOT}, T\}$
 3. (5 points) $\{\text{CNOT}, H\}$
 4. (Bonus: 10 points) $\{\text{CZ}, K, T\}$, where CZ denotes a controlled-Z gate and $K = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$
-

Problem 4. *One-out-of-four search.* Let $f: \{0, 1\}^2 \rightarrow \{0, 1\}$ be a black-box function taking the value 1 on exactly one input. The goal of the one-out-of-four search problem is to find the unique $(x_1, x_2) \in \{0, 1\}^2$ such that $f(x_1, x_2) = 1$.

1. (2 points) Write the truth tables of the four possible functions f .
2. (3 points) How many classical queries are needed to solve one-out-of-four search?
3. (7 points) Suppose f is given as a quantum black box U_f acting as

$$|x_1, x_2, y\rangle \xrightarrow{U_f} |x_1, x_2, y \oplus f(x_1, x_2)\rangle.$$

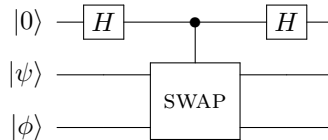
Determine the output of the following quantum circuit for each of the possible black-box functions f :



4. (3 points) Show that the four possible outputs obtained in the previous part are pairwise orthogonal. What can you conclude about the quantum query complexity of one-out-of-four search?
-

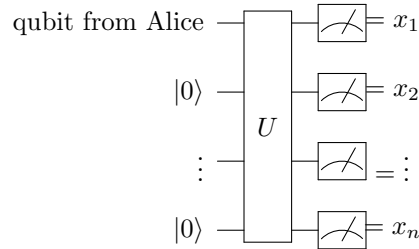
Problem 5. *Swap test.*

1. (5 points) Let $|\psi\rangle$ and $|\phi\rangle$ be arbitrary single-qubit states (not necessarily computational basis states), and let SWAP denote the 2-qubit gate that swaps its input qubits (i.e., $\text{SWAP}|x\rangle|y\rangle = |y\rangle|x\rangle$ for any $x, y \in \{0, 1\}$). Compute the output of the following quantum circuit:



2. (5 points) Suppose the top qubit in the above circuit is measured in the computational basis. What is the probability that the measurement result is 0?
 3. (3 points) If the result of measuring the top qubit in the computational basis is 0, what is the (normalized) post-measurement state of the remaining two qubits?
 4. (2 points) How do the results of the previous parts change if $|\psi\rangle$ and $|\phi\rangle$ are n -qubit states, and SWAP denotes the $2n$ -qubit gate that swaps the first n qubits with the last n qubits?
-

Problem 6. *A qubit cannot be used to communicate a trit perfectly* Suppose that Alice wants to convey a trit of information (an element of $\{0, 1, 2\}$) to Bob and all she is allowed to do is prepare one qubit and send it to Bob. Bob is allowed to prepare $n - 1$ additional qubits, each in state $|0\rangle$, and apply an n -qubit unitary U operation to the entire n qubit system followed by a measurement in the computational basis.



The outcome will be an element of $\{0, 1\}^n$. It is conceivable that such a scheme could exist where Bob can determine the trit from these n bits (e.g., by a function $f(x_1, \dots, x_n) \in \{0, 1, 2\}$). We shall prove that this is impossible.

The framework is that Alice starts with a trit $j \in \{0, 1, 2\}$ (unknown to Bob) and, based on j , prepares a one-qubit state, $\alpha_j|0\rangle + \beta_j|1\rangle, j \in \{0, 1, 2\}$. and sends it to Bob.

Then Bob applies some n -qubit unitary U to $(\alpha_j|0\rangle + \beta_j|1\rangle)|00 \dots 0\rangle$ and measures each qubit in the computational basis, obtaining some $x \in \{0, 1\}^n$ as outcome. Finally, Bob applies some function $f : \{0, 1\}^n \rightarrow \{0, 1, 2\}$ to x to obtain a trit. The scheme *works* if and only if, starting with any $j \in \{0, 1, 2\}$, the resulting x will satisfy $f(x) = j$ with probability 1.

1. (5 points) Note that each row of the matrix U is a 2^n -dimensional vector. For $j \in \{0, 1, 2\}$, define the space V_j to be the span of all rows of U that are indexed by an element of the set $f^{-1}(j) \subseteq \{0, 1\}^n$. Prove that V_0, V_1 , and V_2 are mutually orthogonal spaces.
2. (5 points) Explain why, for a scheme to work, $(\alpha_j|0\rangle + \beta_j|1\rangle)|00 \dots 0\rangle \in V_j$ must hold for all $j \in \{0, 1, 2\}$.
3. (5 points) Prove that it is impossible for $(\alpha_j|0\rangle + \beta_j|1\rangle)|00 \dots 0\rangle \in V_j$ to hold for all $j \in \{0, 1, 2\}$.