Due: October 25th, 2018
Date: October 26, 2018

Assignment 4

Please submit it electronically to ELMS. This assignment is 9% in your total points. For the simplicity of the grading, the total points for the assignment is 90.

Problem 1. *Density matrices.* Consider the ensemble in which the state $|0\rangle$ occurs with probability 3/5 and the state $(|0\rangle + |1\rangle)/\sqrt{2}$ occurs with probability 2/5.

1. *(3 points)* What is the density matrix ρ of this ensemble?
2. *(4 points)* Write ρ in the form $\frac{1}{2}(I + r_x X + r_y Y + r_z Z)$, and plot ρ as a point in the Bloch sphere.
3. *(4 points)* Suppose we measure the state in the computational basis. What is the probability of getting the outcome 0? Compute this both by averaging over the ensemble of pure states and by computing $\mathrm{tr}(\rho|0\rangle\langle 0|)$, and show that the results are consistent.
4. *(4 points)* How does the density matrix change if we apply the Hadamard gate? Compute this both by applying the Hadamard gate to each pure state in the ensemble and finding the corresponding density matrix, and by computing $H\rho H^\dagger$.

Problem 2. *Local operations and the partial trace.*

1. *(3 points)* Let $|\psi\rangle = \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|11\rangle$. Let ρ denote the density matrix of $|\psi\rangle$ and let ρ' denote the density matrix of $(I \otimes H)|\psi\rangle$. Compute ρ and ρ'.
2. *(3 points)* Compute $\mathrm{tr}_B(\rho)$ and $\mathrm{tr}_B(\rho')$, where B refers to the second qubit.
3. *(4 points)* Let ρ be a density matrix for a quantum system with a bipartite state space $A \otimes B$. Let I denote the identity operation on system A, and let U be a unitary operation on system B. Prove that $\mathrm{tr}_B(\rho) = \mathrm{tr}_B((I \otimes U)\rho(I \otimes U^\dagger))$.
4. *(Bonus: 3 points)* Show that the converse of part (c) holds for pure states. In other words, show that if $|\psi\rangle$ and $|\phi\rangle$ are bipartite pure states, and $\mathrm{tr}_B(|\psi\rangle\langle\psi|) = \mathrm{tr}_B(|\phi\rangle\langle\phi|)$, then there is a unitary operation U acting on system B such that $|\phi\rangle = (I \otimes U)|\psi\rangle$.
5. *(Bonus: 2 points)* Does the converse of part (c) hold for general density matrices? Prove or disprove it.

Problem 3. *Product and entangled states.* Determine which of the following states are entangled. If the state is not entangled, show how to write it as a tensor product; if it is entangled, prove this.

1. *(5 points)* $\frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle$
2. *(5 points)* $\frac{1}{2}(|00\rangle - i|01\rangle + i|10\rangle + |11\rangle)$
3. *(5 points)* $\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$

Problem 4. *Working with vectors and operators. (10 points)* Let X and Y be complex Euclidean spaces and let A be a linear mapping (i.e., any non-zero operator) from Y to X. Prove that there exists a complex Euclidean space Z along with vectors $u \in X \otimes Z$ and $v \in Z \otimes Y$ such that

$$A = (I_X \otimes v^*)(u \otimes I_Y),$$

where v^* is the conjugate transpose of v. What's the minimum possible dimension of Z that is required to write a given A in this way?

Problem 5. *Choi-Jamiolkowski representation. (15 points)* Let X, Y, and Z be complex Euclidean spaces, let Φ be a completely positive and trace-preserving channel from X to $Y \otimes Z$ defined as follows:

$$\Phi(A) = (I_Y \otimes \Psi)(\sigma \otimes A), A \text{ is an operator over } X,$$

for some density operator σ over $Y \otimes W$ and a completely positive and trace-preserving channel Ψ from $W \otimes X$ to Z. Prove that there exists a density operator ρ over Y such that

$$\mathrm{Tr}_Z(J(\Phi)) = \rho \otimes I_X,$$

where $J(\Phi)$ is the Choi-Jamiolkowski representation of Φ.

(Hint: consider the proof of the characterization of channels by the Choi-Jamiolkowski representation.)

Problem 6. *Fidelity.* Given any two states ρ, σ over X, let $\mathrm{F}(\rho, \sigma)$ denote the fidelity between ρ and σ.

1. *(10 points)* Find an example of ρ, σ over X, for your choice of a complex Euclidean space X, such that

$$1 - \frac{1}{2}\|\rho - \sigma\|_1 = \mathrm{F}(\rho, \sigma) < \sqrt{1 - \frac{1}{4}\|\rho - \sigma\|_1^2}.$$

   Hint: consider the proof of the Fuchs-van de Graaf inequality, and think about the equality condition.

2. *(15 points)* Define the *fidelity distance* between any two states ρ, σ as

$$d_F(\rho, \sigma) = \min\left\{\||u\rangle - |v\rangle\| : |u\rangle, |v\rangle \in X \otimes Y, \mathrm{Tr}_Y(|u\rangle\langle u|) = \rho, \mathrm{Tr}_Y(|v\rangle\langle v|) = \sigma\right\},$$

   for some complex Euclidean space Y such that $\dim(Y) = \dim(X)$ to allow for the existence of purifications of ρ and σ. Prove that

$$d_F(\rho, \sigma) = \sqrt{2 - 2\mathrm{F}(\rho, \sigma)}.$$

   Also prove that the fidelity distance obeys the triangle inequality:

$$d_F(\rho, \sigma) \leq d_F(\rho, \xi) + d_F(\xi, \sigma),$$

   for any density operator ρ, σ, ξ.