Instructor: Xiaodi Wu
Date: prepared by Xiaodi Wu

Exercises on Linear Algebra (Optional)

Here is a list of example problems on the linear algebra that we will use in quantum information. You **don't** need to submit solutions for these problems, However, you probably want to figure out answers to these problems as well.

Problem 1.1. For complex number $c = a + bi$, recall that the real and imaginary parts of c are denoted $\mathrm{Re}(c) = a$ and $\mathrm{Imag}(c) = b$.

- Prove that $c + c^* = 2 \cdot \mathrm{Re}(c)$.
- Prove that $cc^* = a^2 + b^2$.
- What is the polar form of $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$? Use the fact that $e^{i\theta} = \cos\theta + i \sin\theta$?
- Draw $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$ as a vector in the complex plane.

Problem 1.2. Define that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

- What is $\mathrm{tr}(A |1\rangle \langle 0|)$? (Hint: This can be computed quickly by using the cyclic property of the trace and the outer product representation of A. Do master this trick; it will be used repeatedly in the course and save you much time.)
- Let $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Use the same trick above, along with the fact that the trace is linear, to quickly evaluate

$$\mathrm{tr}(A \cdot |+\rangle\langle+|).$$

Problem 1.3.

- Write out the 4-dimensional vector for $(\alpha |0\rangle + \beta |1\rangle) \otimes (\gamma |0\rangle + \delta |1\rangle)$?
- Let $\mathcal{B}_1 = \{|\psi_1\rangle, |\psi_2\rangle\}$, $\mathcal{B}_2 = \{|\phi_1\rangle, |\phi_2\rangle\}$ be two orthonormal bases for $\mathbb{C}^2$. Can you construct an orthonormal basis for $\mathbb{C}^4$?

Problem 1.4.

- Write out the 4×4 matrix representing $Y \otimes Y$.
- Prove that $(Z \otimes Y)^\dagger = Z \otimes Y$. Do not write out any matrices explicitly; rather, you must use the properties of the tensor product, dagger, and Y.

Problem 2.1.

- Write down a matrix that is not Hermitian.
- Let $A \in \mathrm{L}\left(\mathbb{C}^d\right)$ be a Hermitian matrix. Prove that if for all $|\psi\rangle \in \mathbb{C}^d$, $\langle\psi| A |\psi\rangle \geq 0$, then A has only non-negative eigenvalues.
- Let $A \in \mathrm{L}\left(\mathbb{C}^d\right)$ be a Hermitian matrix. Prove that if A has only non-negative eigenvalues, then for all $|\psi\rangle \in \mathbb{C}^d$, $\langle\psi| A |\psi\rangle \geq 0$.

Problem 2.2. Given $|-\rangle$ state, and suppose that we measure in the computational basis $B = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. What are the probabilities for each possible measurement outcome, and the corresponding post-measurement states?

Problem 2.3.

- Let $A, B \in \mathrm{L}\left(\mathbb{C}^d\right)$ be positive semi-definite matrices. Prove that $A + B$ is positive semi-definite.
- Prove that if ρ and σ are density matrices, then so is $p_1\rho + p_2\sigma$ for any $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$.

Problem 2.4. Let $|\psi\rangle = \alpha_0 |a_0\rangle |b_0\rangle + \alpha_1 |a_1\rangle |b_1\rangle$ be the Schmidt decomposition of a two-qubit state $|\psi\rangle$. Prove that for any single qubit unitaries U and V, $|\psi\rangle$ is entangled if and only if $|\psi'\rangle = (U \otimes V) |\psi\rangle$ is entangled.