Linear Algebra Cheatsheet

1. The state of a system in quantum mechanics are represented by vectors in a (finite dimensional in the case of quantum computing) vector space over the complex numbers. Such spaces are known as Hilbert spaces and the a vector labelled a is denoted by $|a\rangle$. Each vector can be associated with a column vector with complex coefficients, eg. a 4 dimensional Hilbert Space $\mathcal{H}$ can contain vectors such as $\begin{bmatrix} i \\ 0 \\ -\frac{1}{2} \\ \frac{i\sqrt{3}}{2} \end{bmatrix}$

   Hilbert spaces for quantum computing are often of dimension 2^n where n is a positive integer. Any vector in a N dimensional space can be written as the sum of N basis vectors. For example, if we have basis vectors $\{|b_1\rangle, |b_2\rangle, \ldots, |b_N\rangle\}$, an arbitrary vector can be written as $\sum_n c_n |b_n\rangle$

   Often a convenient basis is chosen and termed as the computational basis. The 2^n basis vectors are labelled by the 2^n n bit bitstrings representing the natural numbers from 0 to $2^n - 1$. Here the bit string labelled j has 1 as the j^{th} dimension and 0 everywhere else.

2. The dual of a vector space over the complex numbers is the set of linear maps from vectors in the vector space to the complex numbers. The dual space of a vector space over the complex numbers is itself a vector space over the complex numbers corresponding to row vectors of the same dimension as the original space. The dual space for a Hilbert space $\mathcal{H}$ is $\mathcal{H}^*$. Each column vector in $\mathcal{H}$ is associated with a row vector in $\mathcal{H}^*$, which is its transpose conjugate (obtained by transposing the matrix and taking the complex conjugate of each of its entries) eg. The dual vector of $\begin{bmatrix} i \\ 0 \\ -\frac{1}{2} \\ \frac{i\sqrt{3}}{2} \end{bmatrix}$ is $\begin{bmatrix} -i & 0 & -\frac{1}{2} & \frac{-i\sqrt{3}}{2} \end{bmatrix}$.

   The dual vector of $|\psi\rangle$ is denoted by $\langle\psi|$ (read as a bra).

3. The inner product of two vectors $|\psi\rangle$ with $|\phi\rangle$ is defined as $\langle\phi| |\psi\rangle$ which can be computed by multiplying the row vector representing $\langle\phi|$ with the column vector representing ψ.

   The magnitude of a vector $|\psi\rangle$ is defined as $\sqrt{|\langle\psi| |\psi\rangle|}$

4. The Kronecker delta $\delta_{m'n'}$ is defined to be equal to 1 if $m' = n'$ and 0 otherwise. A set of vectors $\{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\}$ is said to be orthonormal if $\langle b_{m'}| |b_{n'}\rangle = \delta_{m'n'}$. An orthonormal set of basis vectors is called an orthonormal basis. The computational basis introduced earlier is clearly orthonormal.

5. A linear operator on a vector space is a linear transformation $T : \mathcal{H} \to \mathcal{H}$ that maps vectors in $\mathcal{H}$ to vectors in $\mathcal{H}$.

   The outer product of two vectors $|\psi\rangle$ and $|\phi\rangle$ is denoted by $|\psi\rangle\langle\phi|$. The outer product is a linear operator whose action is defined by

   $$(|\psi\rangle\langle\phi|)|\lambda\rangle = |\psi\rangle(\langle\phi| |\lambda\rangle) = (\langle\phi| |\lambda\rangle)|\psi\rangle$$

   The outer product of a vector $|\psi\rangle$ with itself $|\psi\rangle\langle\psi|$ is an operator that maps each vector $|\lambda\rangle$ to $\langle\psi| |\lambda\rangle |\psi\rangle$. Thus the operator acts as a projector projecting vectors into the subspace spanned by $|\psi\rangle$

6. Suppose $B = \{|b_1\rangle, |b_2\rangle, \ldots, |b_N\rangle\}$ is an orthonormal basis. Then every linear operator T on $\mathcal{H}$ can be written as

$$T = \sum_{1 \leq m,n \leq N} T_{mn} \langle b_m| |b_n\rangle$$

(This representation directly encodes the action of T on each basis vector, which linearly extends to other vectors). This also allows us to write each linear operator as a matrix with T_mn as the (m, n) entry.

7. The adjoint of an operator T is denoted by $T^\dagger$ and is defined as that linear operator on $\mathcal{H}^*$ that satisfies

$$(\langle\psi| T^\dagger |\phi\rangle)^* = \langle\phi| T |\psi\rangle, \forall\psi, \phi \in \mathcal{H}$$

8. The identity operator I over a Hilbert Space $\mathcal{H}$ maps every vector in $\mathcal{H}$ to itself. For any orthonormal basis $B = \{|b_i\rangle\}$, $I = \sum_i |b_i\rangle \langle b_i|$

9. A linear operator U with $U^\dagger U = I$ is called a Unitary operator

   A linear operator H with $H = H^\dagger$ is called a Hermitian operator

   A linear operator A with $AA^\dagger = A^\dagger A$ is called a Normal operator (both unitary and hermitian operators are normal)

10. A vector $|\psi\rangle$ is called an eigenvector of an operator T if

$$T|\psi\rangle = c_\psi |\psi\rangle$$

   where c_ψ is a constant and is called the eigenvalue corresponding to $|\psi\rangle$.

   All the eigenvalues of a Hermitian operator are real

11. The trace of an operator T acting on a N-dimensional Hilbert space $\mathcal{H}$ is defined as

$$Tr(T) = \sum_{i=1}^{N} \langle b_i| T |b_i\rangle$$

   where $B = \{|b_i\rangle\}$ is any orthonormal basis. This value is independent of the basis used. It is the sum of the diagoanl entries of the matrix.

12. The following is one of the most important algorithms in linear algebra (and specifically in the context of quantum mechanics)

   Theorem 0.1 (Spectral Theorem)**.** *For every normal operator T over a finite dimensional Hilbert space $\mathcal{H}$ there exists an orthonormal basis for $\mathcal{H}$ that consists of the eigenvectors of T.*

   Let us index the eigenvectors of T as $\{|T_i\rangle\}$, with T_i as the corresponding eigenvalues . T is diagonal in its own eigenbasis and can thus be written as

$$T = \sum_i T_i |T_i\rangle \langle T_i|$$

13. Since $\{|T_i\rangle\}$ is an orthonormal basis, $|T_i\rangle \langle T_i|^m = |T_i\rangle \langle T_i|$. Thus,

$$T^m = (\sum_i T_i |T_i\rangle \langle T_i|)^m = \sum_i T_i^m |T_i\rangle \langle T_i|$$

Thus any function $f : \mathbb{C} \to \mathbb{C}$ with a power series expansion (eg. a Taylor-McLaurin Series) can be extended to linear operators as:

$$f(x) = \sum_j a_j x^j \implies f(T) = \sum_j a_j T^j$$

Thus, rearranging terms, we have,

$$f(T) = \sum_i (\sum_j a_j T_i^j) |T_i\rangle \langle T_i| = \sum_i f(T_i) |T_i\rangle \langle T_i|$$

14. The Tensor product $\otimes$ is used to combine spaces vectors and operators in two Hilbert Spaces $\mathcal{H}_1$ (dimension n) and $\mathcal{H}_2$ (dimension m). Let $\{|b_i\rangle\}$ and $\{|c_j\rangle\}$ be orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then the vectors $\{|b_i\rangle \otimes |c_j\rangle\}$ form the basis for the composite space $\mathcal{H}_1 \otimes \mathcal{H}_2$. For, $|\psi\rangle \in \mathcal{H}_1$ $|\phi\rangle \in \mathcal{H}_2$,

$$|\psi\rangle \otimes |\phi\rangle = \sum_{i,j} \langle b_i| |\psi\rangle \langle c_j| |\phi\rangle |b_i\rangle \otimes |c_j\rangle$$

For operators we require $(A \otimes B)(|\psi\rangle \otimes |\phi\rangle) = A|\psi\rangle \otimes B|\phi\rangle$. Similarly to the definition for vectors we have

$$A \otimes B = \sum_{i,j,i',j'} A_{ij} B_{i'j'} (|b_i\rangle \otimes |c_{i'}\rangle)(\langle b_j| \otimes \langle c_{j'}|)$$