Quantum Error Correction

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(Dated: December 14, 2018)

I. INTRODUCTION

Quantum computers are able to efficiently perform tasks such as solving various optimization problems and simulating physical systems; problems that are believed to be impossible to solve in a reasonable time with classical computers. The road towards building a large-scale quantum computer is, however, hindered by many obstacles, the most significant being the fragile nature of quantum systems.

The delicacy of quantum information makes its storage and manipulation much more challenging than classical information. This, as well as the inability to completely isolate the building-blocks of the quantum computer from its surroundings, introduce errors, which, if are not accounted for, make the computation unfeasible.

In this report, we will survey the field of Quantum Error Correction (QEC). We will begin by discussing in more detail why is QEC necessary and what has to be accomplished. We will present a few examples of error correcting codes such as the surface code [1] and discuss their implementation in current state-of-the-art experiments [2]. Finally we will explain the Quantum Threshold Theorem [3] and how it makes large-scale quantum computing possible. The goal of this paper is to introduce the concepts of QEC and several of its most promising applications.

II. THE QUBIT

Before we delve into the most common types of qubit errors let us briefly review the structure of a qubit. A qubit it a quantum bit, represented by the state:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{1}$$

The qubit can be in the basis states $|0\rangle$, $|1\rangle$, or any linear combination of the two where the complex coefficients obey the normalization relation

$$|\alpha|^2 + |\beta|^2 = 1. \tag{2}$$

This normalization relation for all qubit states means probability must be conserved, and requires all quantum gates to be unitary and reversible, unlike their classical counterparts [4]. Any single qubit gate can be expressed as:

$$A = a_I I + a_x X + a_y Y + a_z Z, \tag{3}$$

where I, X, Y, Z are the 2×2 identity operator and the Pauli operators, respectively. The coefficients in front of these operators also obey a normalization relation:

$$|a_I|^2 + |a_x|^2 + |a_y|^2 + |a_z|^2 = 1. \tag{4}$$

III. FROM CLASSICAL ERRORS TO QUANTUM

Classical bits are represented by $(0), (1)$. A classical error is when the value of the bit unpredictably changes to the other. The simplest classical error correcting code is redundant coding using the majority rule. In this error correcting code, three copies are made of each bit, so that: $(0) \rightarrow (0, 0, 0)$. In this way, it is possible to tell if an error has occurred, since all the bit entries should be the same. Furthermore, once an error has occurred, the majority rule tell us what bit to correct the error back to: what the majority of the bits are. For example, $(0, 1, 1)$ will be corrected to $(1, 1, 1)$[5].

Unlike their classical counterparts, qubits can also have phase errors in addition to bit errors. Phase errors are when a qubit basis state picks up a phase factor, for example, $|0\rangle \rightarrow e^{i\phi}|0\rangle$. These types of errors are significant, because the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is not the same as the state $|\psi\rangle = \alpha|0\rangle + e^{i\phi}\beta|1\rangle$. The angle ϕ can take on any value, so quantum errors are continuous, contrary to the discrete nature of classical ones. This requires a more sophisticated scheme for error detection and correction. QEC schemes must be able to simultaneously correct bit and phase errors.

Unfortunately, the no cloning theorem tells us that unknown quantum states cannot be perfectly copied, making the bit copying schemes that are so commonplace in classical error correction unfeasible. Additionally, we cannot check for errors with direct measurements on the qubit, because doing so would destroy its fragile, coherent state being used in the computation. This is mainly due to the fact that bit and phase errors are detected by making Z and X measurements, respectively. Since $[X, Z] \neq 0$, we know that subsequent measurements of these observable will unintentionally alter the quantum state we are trying to error correct. However, we will show that the redundant coding scheme outlined above does have a quantum analog, which will allow us to side step the non-commuting observable problem. [4].

IV. ERROR MODELS

Before constructing an error correction scheme, it is important to describe what the errors we are trying to mitigate are. In this section, we will introduce two of the most common quantum error sources: coherent quantum errors and environmental decoherence. Coherent quantum errors are when imperfect gates are applied to the qubits. An imperfect knowledge of the dynamics of our quantum two level system that we are using as the our qubit can lead us to use operators (i.e. gates) that aren't doing exactly what we want them to do.

For example, lets say we want to apply the trivial gate I to the state $|0\rangle$. This gate should return the state $|0\rangle$. However, what we believe is I for this system may actually be a rotation about the x-axis on the bloch sphere by a small angle ϵ, we will be left with the state: $\cos(\epsilon)|0\rangle + i\sin(\epsilon)|1\rangle$. So, upon measurement, when we should have a probability of 0 of measuring $|1\rangle$, but due to the coherent gate error, we now will measure $|1\rangle$ with probability $\sin^2(\epsilon)$ [4].

Another common error source arises from some undesired interactions between the system and the environment [4]. This leads to decoherence of the system, which can be described by non-unitary operations. In particular, such a process can turn an initially pure state into a mixed state and has to be described as an operation on density matrices:

$$\rho \to \sum_i E_i \rho E_i^\dagger, \tag{5}$$

where $\sum_i E_i^\dagger E_i = I$. Equation (5) describes the most general error that can occur on a quantum system. Since a density matrix can be thought as an ensemble of pure states $|\psi_j\rangle$, the above is equivalent to applying E_i on each of the $|\psi_j\rangle$, i.e [6]

$$|\psi_j\rangle \to E_i\,|\psi_j\rangle\,, \text{with probability } |E_i\,|\psi_j\rangle|^2. \tag{6}$$

This means that if one can correct all the individual errors E_i, one will also correct the most general error above. Note, however, that E_i is in general not necessarily unitary. As an example, an error occurring on a single qubit is a 2×2 matrix that can be written as a linear superposition of Pauli matrices X, Y, Z and the identity I. It is therefore sufficient to be able to correct each of the errors in the basis $\{X, Y, Z, I\}$ to correct arbitrary single qubit errors.

In general, there can be simultaneous errors occurring on different qubits and these errors could be correlated. An error correcting code that can only correct single qubit errors would fail if two errors were to occur simultaneously on two qubits. If the probability of a single qubit error is ϵ, then the probability of t such simultaneous independent errors would scale as $O(\epsilon^t)$. When ϵ is small it is usually enough to consider codes that can correct up to t errors. This also applies to the case when the errors are correlated; as long as the probability of correlated errors on t qubits also scales as $O(\epsilon^t)$ this will not introduce new problems.

In this report, we will assume that errors on different qubits are independent and any single qubit error is as equally likely as another, i.e X, Y, Z errors all occur with the same probability ϵ. We note that in practice this is a poor assumption, and some errors are more likely than others [6]. Error correcting codes that take this into account can be more efficient than those that do not.

Another type of error that can happen is leakage. This kind of error moves the system outside of the computational Hilbert space and would not be described by the above formalism. While these errors can also be detected and accounted for [7], we will not be discussing them in this report.

V. STABILIZERS

As discussed earlier, quantum errors can be modelled as undesired, random applications of the X, Z operators to the qubit state. One way to detect them is by constantly measuring X and Z. Measurement of a quantum state projects that state onto an eigenstate of the associated measurement operator; destroying the original state and therefore ruining the calculation [2]. Due to the fact that:

$$[X, Z] \neq 0 \tag{7}$$

no state exists that is an eigenstate of both X and Z. In other words, X and Z cannot be known simultaneously for any state.

This problem can be overcome by measuring multiple qubits simultaneously instead of one. For example, say we have two qubits, a and b, with the associated operators X_a, X_b, Z_a, Z_b. We can measure X and Z of each qubit simultaneously without changing the state because these two-qubit operators do commute:

$$\begin{aligned}[X_aX_b, Z_aZ_b] &= (X_aX_b)(Z_aZ_b) - (Z_aZ_b)(X_aX_b)\\ &= X_aZ_aX_bZ_b - Z_aX_aZ_bX_b\\ &= (-Z_aX_a)(-Z_bX_b) - (Z_aX_a)(Z_bX_b) = 0[2]\end{aligned} \tag{8}$$

where we have used the fact that operators acting on different qubits always commute.

The two qubit eigenstates that the operators X_aX_b, Z_aZ_b share are the Bell states:

These operators are examples of **stabilizers**. In general, we say that some state $|\psi\rangle$ is stabilized by operator $\hat{A}$ if $\hat{A}|\psi\rangle = +|\psi\rangle$. By measuring quantum states with stabilizers, the system is kept in a simultaneous eigenstate of the stabilizers, and therefore using them to measure errors will not alter the qubit state.

Z_aZ_b	X_aX_b	$\|\psi\rangle$
+1	+1	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$
+1	-1	$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$
-1	+1	$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$
-1	-1	$\frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$

TABLE I. The Bell States along with their stabilizers and corresponding eigenvalues

We can see that in order to use the stabilizer formalism to error detect, we must use multiple physical qubits to define a single logical one. A logical qubit can be obtained from N physical ones by applying additional constraints to the system, thereby reducing the dimension of the Hilbert space to from 2^N to 2. Going back to our two qubit example with the Bell states, the constraint we might add is that we require the two qubit state $|\psi\rangle$ to be stabilized by Z_aZ_b. Then, we have a logical qubit defined by the two qubit states:

$$|0\rangle_L^{\beta_{00}} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ |1\rangle_L^{\beta_{10}} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle). \tag{9}$$

[4]

We can repeatedly measure X_aX_b, Z_aZ_b, and check the eigenvalues to see if any bit or phase errors have occurred. In this basic example, we demonstrate the power of the stabilizer formalism for error detection. To actually see which qubit the error has occurred on, we need to formulate more complex error correcting codes, which we will introduce in the following sections.

Before we present more complex error correcting protocols, we will discuss another very useful property of the stabilizer formalism. Since we are repeatedly measuring the stabilizers, we are constantly collapsing the multi-physical qubit state into an eigenstate of them. This means that for any arbitrary single qubit error that could change the multi-qubit state in some nontrivial way, the final multi-qubit state will be forced back into an eigenstate of the stabilizers. It is indeed possible that the eigenstate that the corrupted multi-qubit system collapses to is the same as the one before the error even occurred (depending on the error). This means that we won't even detect an error, and the process of detecting an error also corrected it. If we are not so lucky, we can still correct for the error by applying a simple bit or phase flip (X or Z, respectively). The key idea here is that no matter what the single qubit error is, once it is detected we can correct it by applying an X or Z correction gate.

To illustrate this point, consider the following example. Imagine an error represented by the operator $\frac{1}{\sqrt{2}}(I_a+X_a)$ acting on the first qubit in our $|0\rangle_L^{\beta_{00}}$ state:

$$\frac{1}{\sqrt{2}}(I_a + X_a)|0\rangle_L^{\beta_{00}} \\ = \frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)] + \frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)]. \tag{10}$$

After measuring this two qubit system with the stabilizers X_aX_b and Z_aZ_b, we will either get the eigenvalues (Z_aZ_b, X_aX_b)): $(+1,+1)$ or $(-1,+1)$, both with probability $\frac{1}{2}$. In the first measurement outcome, we will have collapsed the state back into the original one, and no correction is necessary. In the latter case, we can correct the error by applying X to either qubit, returning the state back to $|0\rangle_L^{\beta_{00}}$.

VI. ENCODING SCHEMES

We are now ready to discuss how to encode quantum information in a way that is robust to errors; such as the ones introduced in Section IV. As in the classical case, all the codes we will discuss work by enlarging the system and defining a logical qubit in terms of multiple physical qubits.

A. Three-qubit Codes

In this subsection, we will demonstrate encoding of a single logical qubit in terms of three physical qubits. We will discuss the three qubit bit flip code and the three qubit phase flip code.

1. Bit Flips

Recall that a bit flip is defined as:

$$|0\rangle \to |1\rangle\,, |1\rangle \to |0\rangle\,, \tag{11}$$

which corresponds to applying the X operator to a single physical qubit.

A state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ can be encoded into a three qubit state as follows

$$|\psi\rangle \to |\psi\rangle_L = \alpha\,|000\rangle + \beta\,|111\rangle\,, \tag{12}$$

where we identify $|0\rangle_L = |000\rangle$ as the logical "0" state and $|1\rangle_L = |111\rangle$ as the logical "1" state. The logical operations are now defined as: $\bar{X} = X_1X_2X_3$ for an X-gate and $\bar{Z} = Z_1I_2I_3$ for a Z-gate.

The logical qubit defined above can be prepared starting from $|\psi\rangle \otimes |00\rangle$ and utilizing the CNOT gate [4] as shown in the circuit in Fig. 1.

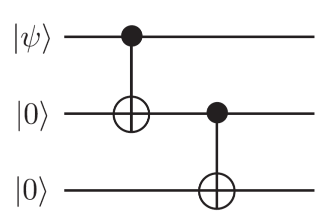

FIG. 1. This quantum circuit creates the logical qubit $|\psi\rangle_L = \alpha\,|000\rangle + \beta\,|111\rangle$. [4]

In the simplest case a bit flip can be detected in one of the three qubits comprising the logical qubit by measuring with each of the following projection operators [8]:

$$\begin{aligned} P_1 &= |000\rangle \langle 000| + |111\rangle \langle 111| \,, \\ P_2 &= |100\rangle \langle 100| + |011\rangle \langle 011| \,, \\ P_3 &= |010\rangle \langle 010| + |101\rangle \langle 101| \,, \\ P_4 &= |001\rangle \langle 001| + |110\rangle \langle 110| \,. \end{aligned} \tag{13}$$

When a bit flip does not occur, $\langle\psi| P_1 |\psi\rangle = 1$, and $\langle\psi| P_{2,3,4} |\psi\rangle = 0$. Likewise, if a bit flip has occurred on qubits 1,2, or 3, then $\langle\psi| P_{2,3,4} |\psi\rangle = 1$, respectively. This means that a measurement of P_1, P_2, P_3, and P_4 tells us which qubit the bit flip occurred on with certainty, or that no error has occurred.

The projection operators do not change the state of the logical qubit; they only decipher which qubit has had a bit flip. The measurements of these operators give no information about the values of α or β. They only give information about which qubit needs to be corrected. Once this information is obtained, the X gate can be applied to the corrupted qubit.

In the stabilizer formalism, there are only two stabilizer operations necessary for detecting bit flips in any one of the three physical qubits comprising the logical qubit. These stabilizers are Z_1Z_2 and Z_2Z_3; where the subscript indicates which qubit has the gate applied to it. Both stabilizers are applied to the logical qubit and the measurement outcome, either +1 or -1, indicates which error has occurred. Once the error is detected, the flipped bit can be corrected if necessary. Table II shows the possible measurement outcomes, the error, and the correction needed to properly implement the error correcting code.

Z_1Z_2	Z_2Z_3	Error	Correction
+1	+1	no error	no action
+1	-1	bit 3 flipped	flip bit 3
-1	+1	bit 1 flipped	flip bit 1
-1	-1	bit 2 flipped	flip bit 2

TABLE II. The three qubit bit-flip code procedure is listed for each of the four possible stabilizer measurement outcomes. [8]

2. *Phase Flips*

A phase flip occurs when the operator Z is applied to a qubit such that the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow Z|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$. Contrary to the bit flip case discussed in the previous section, in this case we encode $|\psi\rangle$ in the following manner:

$$|\psi\rangle \rightarrow |\psi\rangle_L = \alpha|+++\rangle + \beta|---\rangle \,, \tag{14}$$

where we identify $|0\rangle_L = |+++\rangle$ as the logical "0" state and $|1\rangle_L = |---\rangle$ as the logical "1" state. Here, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$. The preparation of the state:$|\psi\rangle_L = \alpha|+++\rangle + \beta|---\rangle$ proceeds identically to the bit flip case with the only difference being the application of Hadamard gates at the end, as shown in Fig. 2. The logical operations are now defined as: $\bar{X} = X_1I_2I_3$ for an X-gate and $\bar{Z} = Z_1Z_2Z_3$ for a Z-gate.

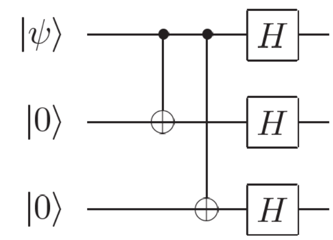

FIG. 2. This quantum circuit creates the logical qubit $|\psi\rangle_L = \alpha|+++\rangle + \beta|---\rangle$. [4]

The new projective operators one needs to measure are obtained from Eq. (13) as $P_j \rightarrow H^{\otimes 3} P_j H^{\otimes 3}$ [8]. This is equivalent to performing the measurements in the Hadamard basis. Explicitly,

$$\begin{aligned} P_1 &= |+++\rangle \langle +++| + |---\rangle \langle ---| \,, \\ P_2 &= |-++\rangle \langle -++| + |+--\rangle \langle +--| \,, \\ P_3 &= |+-+\rangle \langle +-+| + |-+-\rangle \langle -+-| \,, \\ P_4 &= |++-\rangle \langle ++-| + |--+\rangle \langle --+| \,. \end{aligned} \tag{15}$$

As in the bit flip case, measurements of these operators do not give information about the values of α, β. They only tell us which qubit needs a correcting Z gate to be applied to it.

Again paralleling to the bit-flip case, there are two stabilizer operations for detecting phase flips in any one of the three physical qubits comprising the new logical qubit. These stabilizers are X_1X_2 and X_2X_3. Both stabilizers are applied to the logical qubit and the measurement outcomes, either +1 or -1, indicates which error has occurred. Once the error is detected then the bit with the error can be corrected if necessary. Table III shows the possible measurement outcomes, the error, and the correction needed to properly implement the error correcting code.

X_1X_2	X_2X_3	Error	Correction
+1	+1	no error	no action
+1	-1	bit 3 phase flipped	phase flip bit 3
-1	+1	bit 1 phase flipped	phase flip bit 1
-1	-1	bit 2 phase flipped	phase flip bit 2

TABLE III. The three qubit phase-flip code procedure is listed for each of the four possible stabilizer measurement outcomes. [8]

3. Three-qubit Discusion

Encoding a single logical qubit into three physical qubits is used here as an introduction to quantum error correction. Both of the three-qubit codes do not represent a full quantum code. This is apparent once realizing the code cannot simultaneously correct for both bit and phase flips [4].

These codes are an example of repetition code [5] where the system is enlarged such that the logical qubit becomes a state defined by n physical qubits. In the codes above, $n = 3$ and the new logical qubit is more robust demonstrated by the fact that three individual bit flips are required to take $|0\rangle_L$ to $|1\rangle_L$. Thus a single bit flip or phase flip leaves the final state closer to the original undisturbed state.

A crucial limitation of the three-qubit error correcting codes is their inability to correct more than one error. If two qubits experience a bit flip simultaneously then these codes cannot correctly detect both of the errors.

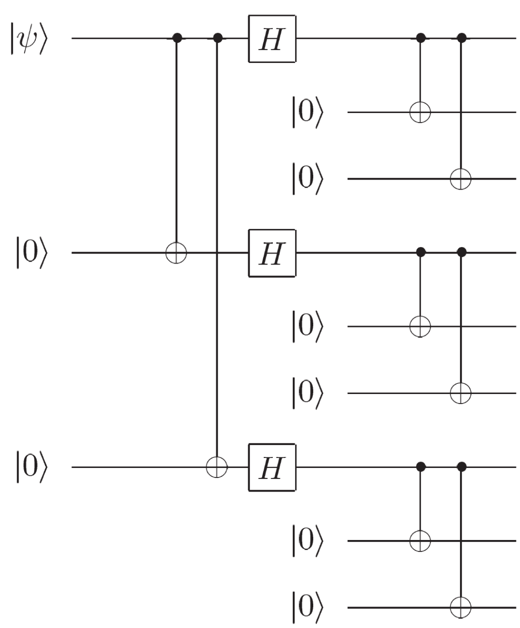

FIG. 3. This circuit encodes the Shor nine qubit code. [8]

B. Shor's nine-qubit code

The Shor code utilizes nine physical qubits to encode a single logical qubit [9] and is a combination of both the 3-qubit bit and phase flip codes discussed in the previous sections. The Shor code is a degenerate single error correcting code; where a degenerate code is capable of correcting more errors than it is able to detect [8]. The Shor code is capable of correcting a logical qubit from one bit-flip, one phase-flip or one of each, on any of the

Name	Stabilizer
g_1	$Z_1Z_2I_3I_4I_5I_6I_7I_8I_9$
g_2	$I_1Z_2Z_3I_4I_5I_6I_7I_8I_9$
g_3	$I_1I_2I_3Z_4Z_5I_6I_7I_8I_9$
g_4	$I_1I_2I_3I_4Z_5Z_6I_7I_8I_9$
g_5	$I_1I_2I_3I_4I_5I_6Z_7Z_8I_9$
g_6	$I_1I_2I_3I_4I_5I_6I_7Z_8Z_9$
g_7	$X_1X_2X_3X_4X_5X_6I_7I_8I_9$
g_8	$I_1I_2I_3X_4X_5X_6X_7X_8X_9$

TABLE IV. The three qubit phase-flip code procedure is listed for each of the four possible stabilizer measurement outcomes. [8]

nine physical qubits [4]. This is the first quantum error correcting code capable for detecting any arbitrary single qubit error.

The encoding of the Shor code can be understood in two parts: the first uses the phase flip code in Eq. (14), namely, $|\psi\rangle = \alpha\,|+++\rangle + \beta\,|---\rangle$.

The second part consists of encoding each of the three original qubits into three more qubits using the bit flip code, namely $|+\rangle \rightarrow |000\rangle + |111\rangle$ and $|-\rangle \rightarrow |000\rangle - |111\rangle$. The preparation circuit is shown in Fig. 3.

The end result corresponds to the nine-qubit encoding, $|\psi\rangle = \alpha\,|0\rangle_L + \beta\,|1\rangle_L$, where:

$$\begin{aligned}|0\rangle_L &= \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle),\\ |1\rangle_L &= \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle).\end{aligned} \tag{16}$$

The logical operations on the new qubit are: $\bar{X} = Z_1Z_2Z_3Z_4Z_5Z_6Z_7Z_8Z_9$ for an X-gate and $\bar{Z} = X_1X_2X_3X_4X_5X_6X_7X_8X_9$ for a Z-gate.

To summarize, the Shor code is the concatenation of the bit flip and phase flip error correcting codes. The fault-tolerant technique called concatenation will be discussed in more detail later in Section VII A. Like the bit and phase flip error correcting codes there are stabilizers for the Shor code. However, this time, there are eight total stabilizers which can detect and diagnosis errors listed in table IV.

The simplest way to think about the Shor code is to break up the state $|\psi\rangle$ into blocks of three qubits; so, as an example, $|0\rangle_L$ would have three blocks containing $(|000\rangle + |111\rangle)$. The typical procedure of error correction begins by applying the first six stabilizers each of the three blocks to detect bit flip errors. There are two bit flip stabilizers for each block. After correcting any bit flip error, the last two stabilizers are applied to each block to detect phase flip errors.

To further understand Shor's code lets follow an example where the forth physical qubit suffers from both a bit and phase flip error such that the new encoded state is described by: $|\psi\rangle = \frac{\alpha}{2\sqrt{2}}(|000\rangle + |111\rangle)(|100\rangle - |011\rangle)(|000\rangle +$

$|111\rangle) + \frac{\beta}{2\sqrt{2}}(|000\rangle - |111\rangle)(|100\rangle + |011\rangle)(|000\rangle - |111\rangle)$.

The bit flip stabilizers are first applied to each three qubit block. All stabilizers will return +1 as the outcome of measurement except for g_3, which will return a -1 when applied to the second block of physical qubits. So we conclude that the error must be in the forth qubit and apply a correction gate of $I_1I_2I_3X_4I_5I_6I_7I_8I_9$.

After correcting the bit flip, the state is: $|\psi\rangle = \frac{\alpha}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle + |111\rangle) + \frac{\beta}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)$. The phase flip stabilizers, that is g_7 and g_8, are now applied to each block of physical qubits. The measurement results in -1 in both cases indicating a physical qubit in the second block has flipped. Applying a Z-gate to any or all qubits in the second block works for correction.

C. Stabilizers in Practice: The Surface Code

Surface codes utilize the stabilizer formalism for constructing a large scale quantum processor. The surface code is a 2D array of physical qubits that defines a single logical one. Surface codes have a number of advantages, such as a higher single qubit error tolerance and circuit scalability. These advantages do come with a tradeoff; a logical qubit with a reasonably high error threshold (about 1%) needs on the order of 10^3 to 10^4 physical qubits to define it [2].

The 2D array of physical qubits is composed of two different qubit types: data qubits and measure qubits. As their names suggest, data qubits store and manipulated quantum information, while the measure qubits measure them. The measure qubits are where the stabilizer formalism comes into play. These measure qubits are specially designed to measure the stabilizer operators for this system.

There are two types of the measure qubits: X-syndrome and Z-syndrome. X-syndrome measure qubits will measure the X operator for a given data qubit, which means that they detect phase errors. The Z-syndrome measure qubits measure the Z operator for a given data qubit, thereby detecting bit errors. In Fig. 4, the X- and Z-syndromes measure the stabilizers $X_aX_bX_cX_d$ and $Z_aZ_bZ_cZ_d$, respectively with their corresponding quantum circuits shown. Table V displays the eigenvalues and eigenstates of these stabilizers.

The measure qubits measure these stabilizers repeatedly, forcing the N physical qubit system into some eigenstate of the stabilizers. When an X- or Z- syndrome measure qubit detects an unwanted change in an eigenvalue from a previous measurement, we know an error has occurred. Furthermore, we can isolate which physical data qubit the error occurred on, since two of each syndrome measure qubit is coupled to the data qubits. For example, if a data qubits suffers from a bit flip, the two Z-syndrome qubits that are coupled to that data qubit will detect a change in sign of the eigenvalue, and the appropriate error corrections can be applied to the qubit.

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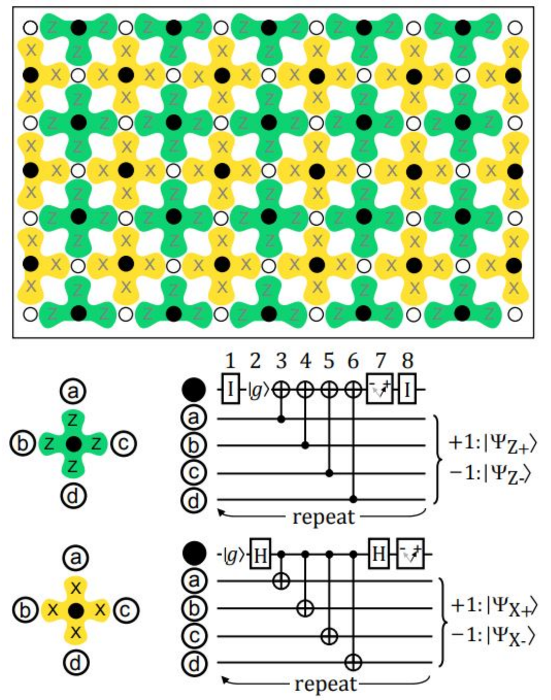

FIG. 4. Schematic of the 2D array of physical qubits that is the surface code. Shaded and unshaded dots are measure and data qubits, respectively. The measure qubits with yellow crosses are X-syndrome and the ones with green crosses are Z-syndrome. Excluding the edges, all data qubits are coupled to two X- and Z- syndromes. Below are the measurement protocols for each type of data qubit. [2].

VII. FAULT-TOLERANT QUANTUM COMPUTATION

Thus far in this report we have concentrated on how to protect individual qubits by encoding them into "codewords". Running a quantum algorithm *fault-tolerantly* on those codewords nevertheless requires us to consider additional challenges. Amongst those challenges are the fact that we need to be able to initialize the codewords, measure the stabilizers and apply corrections, if necessary, and all those operations could fail. Another potential problem lies in the fact that every code has a finite distance d, which specifies the number of error it can correct, given by $\lfloor \frac{d-1}{2} \rfloor$.

This has several consequences: first, for long enough quantum circuits, we are essentially guaranteed that unrecoverable errors will occur; second, gates and other operations can propagate errors, and in particular, they can increase the weights of the errors, making them uncorrectable with a given code. To illustrate the second point, consider applying a unitary U on a state $|\psi\rangle$ with a pre-existing error E [6]:

$$UE\,|\psi\rangle = (UEU^\dagger)U\,|\psi\rangle\,. \tag{17}$$

Assuming that the unitary is perfect, we can view the resulting state as the correct state $U\,|\psi\rangle$ with an error $UEU^\dagger$. If U is a single qubit gate this simply changes the type of error. For instance, a X error can become Z, which is not problematic. However, if U is a two qubit gate, a single qubit error E can become a two qubit error $UEU^\dagger$. If we use a code with distance 3 which can correct single qubit errors, by applying U we have turned a correctable error to an uncorrectable error. The solution to this problem consists of cleverly designing the gates to avoid the propagation of errors. While it is impossible to design a universal gate set that does not spread errors at all [10], there are ways of making the spreading manageable and the errors correctable [6].

Eigenvalue	$\hat{Z}_a\hat{Z}_b\hat{Z}_c\hat{Z}_d$	$\hat{X}_a\hat{X}_b\hat{X}_c\hat{X}_d$
$+1$	$\|gggg\rangle$	$\|++++\rangle$
	$\|ggee\rangle$	$\|++--\rangle$
	$\|geeg\rangle$	$\|+--+\rangle$
	$\|eegg\rangle$	$\|--++\rangle$
	$\|egge\rangle$	$\|-++-\rangle$
	$\|gege\rangle$	$\|+-+-\rangle$
	$\|egeg\rangle$	$\|-+-+\rangle$
	$\|eeee\rangle$	$\|----\rangle$
-1	$\|ggge\rangle$	$\|+++-\rangle$
	$\|ggeg\rangle$	$\|++-+\rangle$
	$\|gegg\rangle$	$\|+-++\rangle$
	$\|eggg\rangle$	$\|-+++\rangle$
	$\|geee\rangle$	$\|+---\rangle$
	$\|egee\rangle$	$\|-+--\rangle$
	$\|eege\rangle$	$\|--+-\rangle$
	$\|eeeg\rangle$	$\|---+\rangle$

TABLE V. The stabilizers that the X- and Z- syndrome qubits measure, along with their corresponding eigenvalues and eigenstates. [2]

A. Concatenation

One way of making a quantum circuit more robust to errors is by *concatenating* quantum error-correcting codes to form new, larger codes. Suppose that we have independent single qubit errors that occur with probability p and assume for simplicity that we have a code of distance 3 that can correct them. The remaining, unrecoverable errors will have a probability of cp^2 for some constant c, to lowest order in p [11]. Note that the code leads to an improvement, i.e reduces the error probability from p to cp^2, provided that $p < 1/c$ where $1/c \equiv p_{th}$ is the error threshold probability. In general, p_{th} depends on the quantum error correcting code, the error model and other details such as the connectivity graph between qubits.

The basic idea behind concatenating codes is illustrated in Fig. 5. At the first level, we encode every qubit with some code. At the second level, each qubit forming the codeword gets encoded again, using the same or a different code. If we use the same n-qubit code, after two levels, each logical qubit is effectively encoded by n^2 physical qubits. For k encoding levels, a logical qubit is formed by n^k physical qubits.

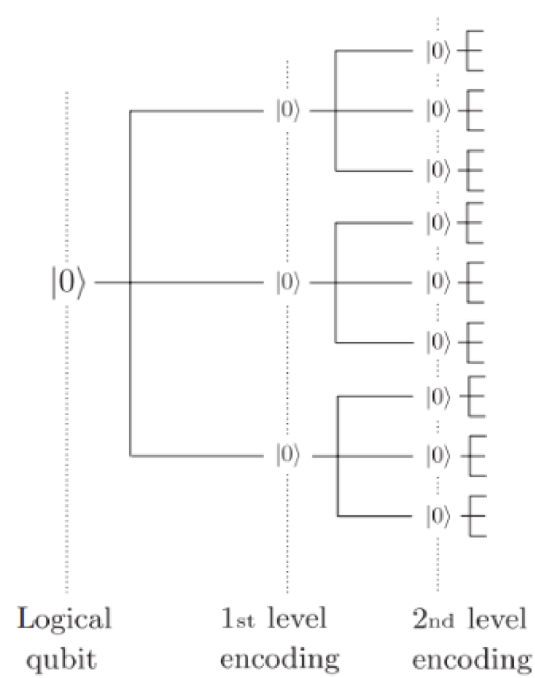

FIG. 5. Schematic illustrating concatenation of codes [11].

This procedure allows us to make the error probability in the logical qubits arbitrary small. Recall that at the level of the physical qubits we had errors with rate p. At the next encoding level, the error probability is cp^2 and for each subsequent level, the error decreases by a factor of cp^2. Therefore, if we concatenate k times, the error rate on the logical qubits is $c^{-1}(cp)^{2^k} = p_{th}(p/p_{th})^{2^k}$. Thus, provided that $p < p_{th}$, we can make the error rate of the logical qubits arbitrary small, simply by increasing k. This result serves as the basis for the quantum threshold theorem, discussed in the next section (the presentation of the threshold theorem follows chapter 10.6 of [11].)

B. Quantum Threshold Theorem

Concatenating error-correcting codes allows us to implement a quantum circuit with arbitrary small error ϵ while the size of the circuit increases polynomially with $\log \epsilon$. Intuitively, one can see this from the results derived in the previous section: the number of physical qubits scales exponentially with k, the number of concatenations, but the error on the logical qubits decreases as a double exponential.

We assume that we are able to perform operations (gates, measurements) fault-tolerantly. More precisely, an implementation of a gate acting on an k-encoded state is said to be fault-tolerant if the probability of introducing unrecoverable errors during the implementation is smaller than $p_{th}(p/p_{th})^{2^k}$.

Given this, requiring that a fault-tolerant implementation of a circuit of size S (S is the number of logical gates) only fail with a probability of ϵ yields the following condition

$$S p_{th}(p/p_{th})^{2^k} < \epsilon. \tag{18}$$

Let us assume that a fault-tolerant implementation of every gate can be accomplished by at most d^k physical gates, for some constant d. Our goal is to find a bound on d^k which will show that the error rate decreases faster than the increase in the physical circuit size.

Taking the log of both sides of Eq. (18) and rearranging we find

$$2^k < \frac{\log\left(\frac{S p_{th}}{\epsilon}\right)}{\log\left(\frac{p_{th}}{p}\right)}. \tag{19}$$

Since $(2^k)^{\log_2 d} = d^k$, Eq. (19) can be rewritten as

$$d^k < \left(\frac{\log\left(\frac{S p_{th}}{\epsilon}\right)}{\log\left(\frac{p_{th}}{p}\right)}\right)^{\log_2 d}. \tag{20}$$

If we are only interesting on the dependence on S and ϵ, Equation (20) says that

$$S d^k = O\left(S \log^m \left(\frac{S}{\epsilon}\right)\right), \tag{21}$$

i.e the size of fault-tolerant circuit (the number of physical gate Sd^k) grows as polynomial of power $m = \log_2 d \geq 1$ in $\log(S/\epsilon)$.

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