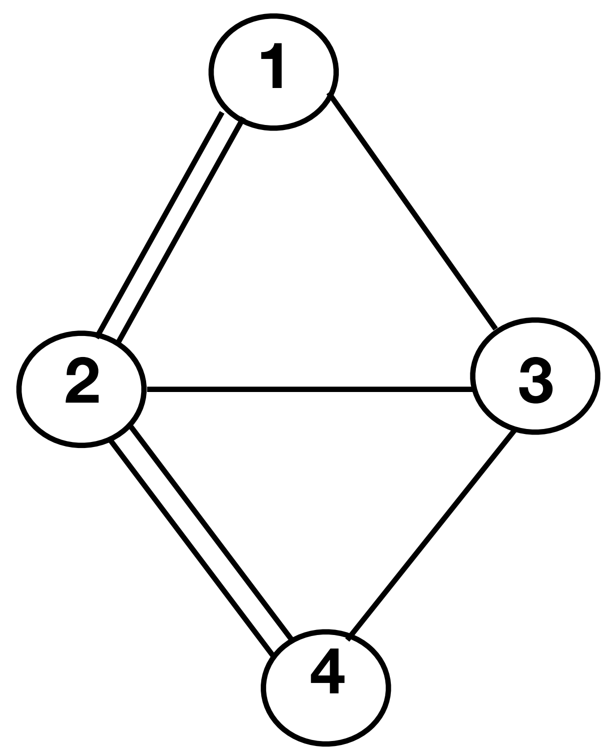
# Eulerian Paths and Cycles

- Start from any vertex
- Traverse each edge once and only once without lifting your pencil from the paper
- You may visit a vertex more than once but not an edge.



# Konigsberg


MONS REGIVS
F I V
B R E
G
E L A
MONS REGIVS, PRVSIÆ SIVE BORVSSIÆ, VRBS MARITIMA ELEGÃTIS-SIMA PRINCIPIS SEDES

# Konigsberg Cartoon

# Eulerian Cycle - Undirected Graph

- Theorem (Euler 1736)
  Let G = (V,E) be an undirected, connected graph. Then G has an Eulerian cycle iff every vertex has an even degree.

  Proof 1: Assume G has an Eulerian cycle. Traverse the cycle removing edges as they are traversed. Every vertex maintains its parity, as the traversal enters and exits the vertex, since exactly two edges are removed. (There is a small special case for the first vertex, which changes its parity at the very beginning and at the the very end.) When done, every vertex must have degree 0, so every vertex must have started with even degree.

# Eulerian Cycle - Undirected Graph

- Theorem (Euler 1736)
  Let G = (V,E) be an undirected, connected graph. Then G has an Eulerian cycle iff every vertex has an even degree.

  Proof 2: Assume every vertex has even degree. Start at an arbitrary vertex and traverse edges, removing them from the graph, until getting stuck at the original vertex. The path can only get stuck at the original vertex since every vertex has even degree, so the path must be a cycle (although not necessarily an Eulerian cycle).
  If it is not an Eulerian cycle, what is left from the original graph are some connected components, each connected to the cycle (possibly at more than one vertex). Find a vertex on the cycle that still has edges. Find a cycle from that vertex, as done above. SPLICE the two cycles together.
  Continue it this fashion until there is only one cycle, which will be an Eulerian cycle.

  END PROOF

  NOTE: The proof actually gives an algorithm for finding the cycle. With a little care, the algorithm is linear time.

# Eulerian Path - Undirected Graph

- Theorem (Euler 1736)
  Let G = (V, E) be an undirected, connected graph. Then G has an Eulerian path iff every vertex, except possibly two of them, has even degree.

  Proof: Basically the same proof as above, except when producing the path start with one vertex with odd degree. The path will necessarily end at the other vertex of odd degree. At that point, the proof is exactly the same as above.

  END PROOF

# Eulerian Cycle and Path - directed Graph

- Theorem (Euler 1736)
  Let G = (V,E) be a directed, strongly connected graph. Then G has an Eulerian cycle iff every vertex has indegree equal to its outdegree.

- THEOREM (Euler 1736)
  Let G = (V,E) be a directed, strongly connected graph. Then G has an Eulerian path iff every vertex has indegree equal to its outdegree, except possibly two of them, one of which has indegree one greater than outdegree, and one of which has outdegree one greater than indegree.