MATH299M/CMSC389W
Fall 2019 – DAn Zou, Devan Tamot, Vlad Dobrin
Project 1: Visualizing Taylor Approximations
Assigned: Friday September 22^{rt}
Due: Monday October 14^{th}, 11:59PM

Welcome to your first project! What I would like you to do is to construct a model that shows the Taylor Approximation of an arbitrary function up to an arbitrary number of steps. How you implement this and what features you add are up to you. The final product should be cleaned up, that is, there should be a title, your name and the date on it, text in Text cells, and no extraneous code or outputs floating around. After making your model, you should play around with it, and maybe in a Text cell describe some interesting behavior you found. Following is an explanation of Taylor Approximations.

The N^{th} term in the *Taylor Expansion* of a function together with the preceding N terms (the "first" term is the 0^{th} term when N=0) makes the N^{th} order *Taylor Approximation* of a function. This is the N^{th} order polynomial that fits the function best. The full infinite sum is called a *Taylor Series.* Taylor Approximations are done around a *center*, denoted a, which is the point the approximation is around. When a=0 we call this a *Maclaurin Series*. This is what the n^{th} term of the Taylor expansion centered at a of a function f(x) is:

$$f^{(n)}(a)\frac{(x-a)^n}{n!}$$

Note that $f^{(n)}(a)$ is notation for $\frac{d^n f(a)}{dx^n}$.
Note that for $\frac{d^n f(a)}{dx^n}$ you take the n^{th} derivative of f wrt x first, then substitute x=a.

The full N^{th} order Taylor Polynomial of a function centered at a is then:

$$\sum_{k=0}^{N} f^{(k)}(a)\frac{(x-a)^k}{k!}$$

The 0^{th} order Taylor Approximation is the best constant approximation we can make to a function at a certain point. According to our formula, this means that we're using the "0^{th} derivative of f", which is just the function itself, evaluated at the center a. This is of course just the horizontal line f(a). Without any *derivatives* to work with, we don't know anything about how the function *changes*, which is exactly what a derivative is. This, then, is the best we can do.

If we look at the 1^{st} order Taylor Approximation, we use the 1^{st} derivative of f(x) at a, which is the slope of the function at x=a. This together with f(a) itself gives us the materials we need to create a linear approximation in the familiar form y=mx+b; this approximation is a line that intersects f(x) at x=a and at that point shares its slope.

Now, if we take the 2^{nd} order Taylor Approximation, we have the 2^{nd} derivative of f(x) to work with. Recall from calculus that the *2^{nd} Derivative Test* allows one to determine the concavity of a function at a certain point. More generally, we can think of the scalar on the quadratic term of a polynomial to be the *concavity* of that function – if it is positive, we are dealing with a concave up function, if it is negative, concave down, and 0 means it has no curvature there. Furthermore, a higher positive concavity means the function is a tighter upwards cup, and a lower negative concavity means it is a tighter downwards cup. Combining the 0^{th}, 1^{st} and 2^{nd} Taylor Approximation terms, we construct a quadratic function ax^2+bx+c that matches the function's value at a, its slope at a, and its concavity at a.

At this point the mechanism of the Taylor Approximation should be coming into focus – by matching a function at a point up to higher and higher derivatives, we create a polynomial with more and more knowledge of how the function is changing, allowing for a better and better approximation of the nearby points. In principal, all of the information of a function is carried in a single point, if you look at all infinitely many of its derivatives, that is, f(a), f'(a), f''(a), f'''(a),… Integrating the N^{th} derivative of a function will give you the $(N-1)^{th}$ derivative plus a constant term; by using all N derivatives, we can fill in those constant term gaps. This was alluded to above for the low-N cases; the 2^{nd} derivative of a function only gives you its concavity, and accordingly the 2^{nd} integral of f''(x) gives you a function in the form $ax^2 + c_1x + c_2$ where a is known but c_1 and c_2 are unknown constants of integration. We fix these free variables by using the first derivative to find the slope c_1 and the function itself to find the constant term c_2.

In this sense the N^{th} Order Taylor Approximation is the best polynomial approximation to a function at a given point. This remarkable tool's power lies in the fact that if the values of all of the derivatives of a function are known at a single point, we can integrate them all to retrieve the design of the entire function. In fact, you can construct the terms of the Taylor Approximation yourself – try taking the N^{th} derivative of the general form of the N^{th} order Taylor Polynomial; what do you get back?