Today, we will consider three search trees, which outwardly look different, but all are equivalent (or nearly so)

All support find, insert, and delete in \(O(\log n)\) time for a tree with \(n\) nodes

These are:
- 2-3 Trees
- Red-black Trees
- AA Trees
2-3 Tree
A Variable Width Tree

- **2-Node:**
  - Two children; stores one key; order: $A < b < C$

- **3-Node:**
  - Three children; stores two keys; order: $A < b < C < d < E$
2-3 Tree

Formal Definition

- A **2-3 tree** is:
  - An empty tree (i.e., null)
  - Root is a 2-node and two subtrees are 2-3 trees of **equal** height
  - Root is a 3-node and its three subtrees are 2-3 trees of **equal** height

- **Theorem**: A 2-3 tree with \( n \) nodes has height \( O(\log n) \)

- **Proof**: (Easy) The sparsest tree is already a complete binary tree
2-3 Tree Insertion

- **Start as usual**: Find the key and note the leaf node where we **fall out** of the tree.
- **Insert new key** in this leaf, and **restructure** if needed:
  - 2-node $\rightarrow$ 3-node - No problem
  - 3-node $\rightarrow$ 4-node - !!
    - Split into two 2-nodes; **promote** middle key to parent; $4 = 2 + 2$
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    - May need to fix parent or create a new root

![Diagram of 2-3 tree insertion](image)
2-3 Tree Insertion

- Example:
2-3 Tree Deletion

- Deletion as usual:
  - Find the key
  - If it is not a leaf, find the replacement node (inorder successor)
  - Copy replacement-node contents to deleted node
  - Recursively delete the replacement node
  - (We may assume that restructuring always starts at the leaf level)
2-3 Tree Deletion

- **Restructuring:**
  - 3-node → 2-node: No problem
  - 2-node → 1-node: Two possible fixes:
    - Adopt from sibling
    - Merge with sibling

- **Adoption:**
  - If there is a 3-node sibling
  - Adopt its closest subtree
  - ...and associated key
  - $1 + 3 = 2 + 2$
2-3 Tree Deletion

- **Restructuring:**
  - 3-node $\rightarrow$ 2-node: No problem
  - 2-node $\rightarrow$ 1-node: Two possible fixes:
    - Adopt from sibling
    - Merge with sibling

- **Merging:**
  - No sibling is 3-node $\Rightarrow$ 2-node
  - Merge these nodes: $1 + 2 = 3$
  - Demote key from parent
  - May need to fix parent or delete root
2-3 Tree Deletion

- Example:
Red-Black Trees

- 2-3 trees are **not** binary trees - Can we simulate the same idea as a binary tree?
- Replace each 3-node with a pair of nodes:
  - To distinguish them, we’ll color the upper node black and the lower node red
  - The result is called a **Red-Black Tree**
Red-Black Trees

- 2-3 trees are **not binary trees** - Can we simulate the same idea as a binary tree?
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Red-Black Trees

Definition:

- Each node is either red or black
- The root is black
  - This corresponds to the fact that the root is either a 2-node or the first half of a 3-node
- All null pointers are considered black
  - This is just a convenient convention
- If a node is red, then both its children are black
  - This enforces the condition that a child of the second half of a 3-node [red] must either be a 2-node [black] or the first half of a 3-node [black]
- Every path from a given node to any of its null descendants contains the same number of black nodes
  - This corresponds to the requirement that all leaves of the 2-3 tree are of equal depth
Red-Black Trees

Lemma: Every 2-3 tree corresponds to a red-black tree

- But the **converse does not hold**. There are valid red-black trees that are not the encoding of some 2-3 tree
- (a) The red child could be on either the **left** or **right** side
- (b) **Both children** of a black node may be **red**

In fact, red-black trees are a binary encoding of a more general tree, a **2-3-4 tree**
AA Trees

- A simpler variant of the red-black tree
- Invented by Arne Anderson (1993) to simplify coding of red-black trees
- Updated definition:
  - If a node is red, then both its children are black
  - It is the right child of a black node
- This fits exactly with our encoding of 2-3 trees as binary trees
AA Trees

Node Representation:

- **No null pointers**: Use a *sentinel node*, called `nil`.
  
  ```
  nil.left = nil.right = nil
  ```

  Reduces need for checking null pointers.

- **No node colors**: Every node stores a *level number*:
  - `nil` is at level 0
  - Leaves at level 1
  - If you are a *red node*, you are at the same level as your parent
  - If you are a *black node*, you are at one level less than your parent
  - Levels match levels of 2-3 tree
AA-Trees
Restructuring

- **skew(p):** If p is black and has a red left child, rotate so that the red child is now on the right

- **split(p):** If p is black and has a right chain of two consecutive red nodes, split this triple, promoting p's right child to the next higher level
AA Trees

Restructuring Operations

```c
AANode skew(AANode p) {
    if (p.left.level == p.level) {  // red node to our right?
        AANode q = p.left;          // do a right rotation at p
        p.left = q.right;           // return pointer to new upper node
        q.right = p;
        return q;
    } else return p; // else, no change needed
}

AANode split(AANode p) {
    if (p.right.right.level == p.level) { // right-right red chain?
        AANode q = p.right;         // do a left rotation at p
        p.right = q.left;
        q.left = p;
        q.level += 1;               // promote q to next higher level
        return q;                   // return pointer to new upper node
    } else return p; // else, no change needed
}
```
AA Trees - Insertion

Insertion

- Search for the new key and note where we fall out of the tree
- Insert a new (red) leaf node here (at level 1)
- Work back towards the root and **restructure** along the way
  - Left child is red? → skew
  - Two red children to the right? → split
AA Trees

Insertion

```c
AANode insert(Key x, Value v, AANode p) {
    if (p == nil)                           // fell out of the tree?
        p = new AANode(x, v, 1, nil, nil);  // ... create a new leaf node here
    else if (x < p.key)                     // x is smaller?
        p.left = insert(x, v, p.left);      // ...insert left
    else if (x > p.key)                     // x is larger?
        p.right = insert(x, v, p.right);    // ...insert right
    else
        throw DuplicateKeyException;        // duplicate key!
    return split(skew(p));                  // restructure and return result
}
```

Only difference with standard binary search tree insertion
AA-Trees

Insertion Example

1. **Insertion Example**

2. **Diagram of AA-Trees**

   - **Insertion** of 6
   - **Skew** operation
   - **Split** operation
AA-Trees - Deletion

- Find the node to delete
- If it is not a leaf, find replacement at the leaf level and delete replacement
- Work back towards the root and restructure along the way
  - More cases than with insertion
  - Basic issue is that a node’s level may decrease

- Possibly 3 skew invocations:
  - skew(p), skew(p.right), skew(p.right.right)

- Possibly 2 split invocations:
  - split(p), split(p.right)
AA Trees

Deletion - Restructuring Utilities

```c
AA_Node updateLevel(AA_Node p) {
    int idealLevel = 1 + min(p.left.level, p.right.level);
    if (p.level > idealLevel) {
        p.level = idealLevel;
        if (p.right.level > idealLevel)
            p.right.level = idealLevel;
    }
    return p;
}

AA_Node fixupAfterDelete(AA_Node p) {
    p = updateLevel(p);
    p = skew(p);
    p.right = skew(p.right);
    p.right.right = skew(p.right.right);
    p = split(p);
    p.right = split(p.right);
    return p;
}
```
AA Trees - Deletion

```java
AANode delete(Key x, AANode p) {
    if (p == nil)                               // fell out of tree?
        throw KeyNotFoundException;             // ...error - no such key
    else {
        if (x < p.key)                          // look in left subtree
            p.left = delete(x, p.left);
        else if (x > p.key)                     // look in right subtree
            p.right = delete(x, p.right);
        else {                                  // found it!
            if (p.left == nil && p.right == nil)// leaf node?
                return nil;                     // just unlink the node
            else if (p.left == nil) {           // no left child?
                AANode r = inorderSuccessor(p); // get replacement from right
                p.copyContentsFrom(r);          // copy replacement contents here
                p.right = delete(r.key, p.right);// delete replacement
            }
            else {                              // no right child?
                AANode r = inorderPredecessor(p);// get replacement from left
                p.copyContentsFrom(r);          // copy replacement contents here
                p.left = delete(r.key, p.left); // delete replacement
            }
        }
    }
    return fixupAfterDelete(p);             // fix structure after deletion
}
```
AA-Trees

Deletion Example
Summary

- 2-3 Trees
- Insertion
  - Splitting nodes
- Deletion
  - Adoption
  - Merging
- Red-black trees - Model 2-3-4 trees
- AA trees - Simplified red-black trees
  - Skew and split to restructure