CMSC 420 – 0201 – Fall 2019
Lecture 08

Splay Trees

Recap

We have seen many variants on the binary search tree

- (Standard) Binary search trees: No balance. O(log n) height/time if operations are random
- AVL trees: A classic, height-balanced binary tree. O(log n) performance guaranteed. Good, but not the fastest in practice
- 2-3 trees: A tree that allows nodes to have 2 or 3 children. O(log n) performance guaranteed. Some space wastage
- Red-black trees: A binary implementation of 2-3 (actually 2-3-4) trees. O(log n) performance guaranteed. Considered among the fastest deterministic structures
- AA trees: A kinder, simpler red-black tree
- Treap and Skiplists: Randomized search structures. O(log n) performance in expectation (over random choices). Very simple and practical

Recap

Are we done yet?

- There are still many interesting extensions
  - Order-statistic queries: Find the kth smallest key
  - Range queries: Count/sum/report all the keys in the interval $[x_0, x_1]$
  - Split/Merge: Given a tree T and key x, split T into subtrees T_1 and T_2, such that keys in T_1 are at most x, and keys in T_2 are greater than x. Merge reverses this, melding two trees (with one having keys smaller than the other) into a single tree
  - Expected-case Optimal Trees: Given access probabilities for the elements, build a tree that minimizes the expected search time. (Static optimality)

Recap

Optimal Binary Search Trees

- Optimal Search Trees:
  - Let $\{x_1, \ldots, x_n\}$ be the keys
  - Let p_i denote the access probability of x_i. Where, $0 \leq p_i \leq 1$, and $p_1 + \cdots + p_n = 1$.
  - High-probability items should be stored near root
  - Can be solved by dynamic programming
- Static optimality: We assume that access probabilities never change
- Dynamic optimality?
  - Suppose that access probabilities do change.
  - Can we build a tree that automatically adjusts to the current distribution?
  - Yes! Splay trees

Splay Trees

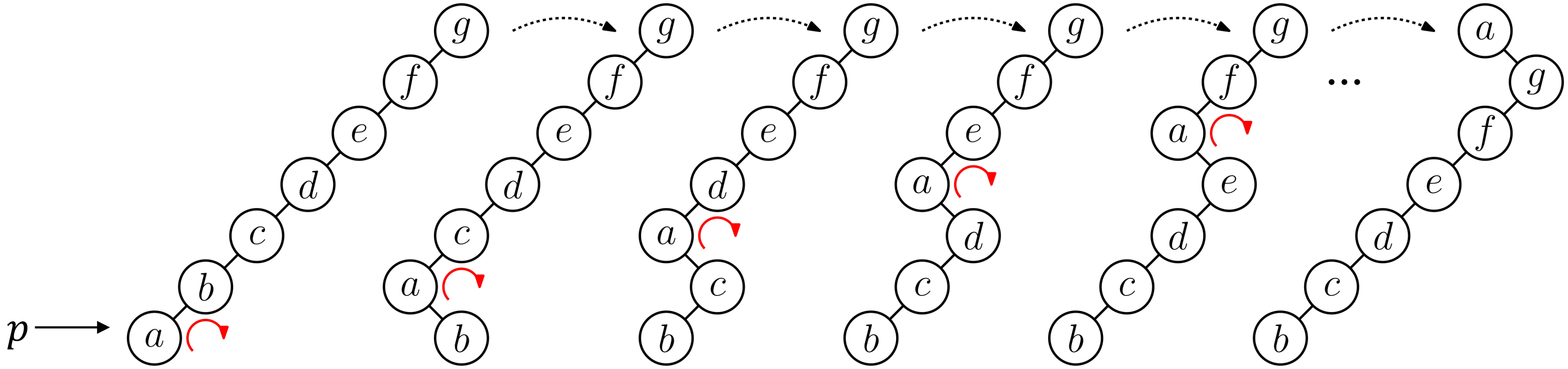
Intuition

- We seek a tree structure that readjusts itself, depending on the access pattern
- Want low-probability nodes near the bottom and high-probability nodes near top
- Intuition:
  - Keys near the bottom of long access chains have high cost
  - Whenever we access a key, let's pull it up to the root
  - Frequently accessed keys will tend to "rise to the top," leading to faster access and better expected performance
- But how do we pull a node up to the root?
  - Need to preserve inorder structure – use rotations!

Splay Trees

A good idea, that doesn't work

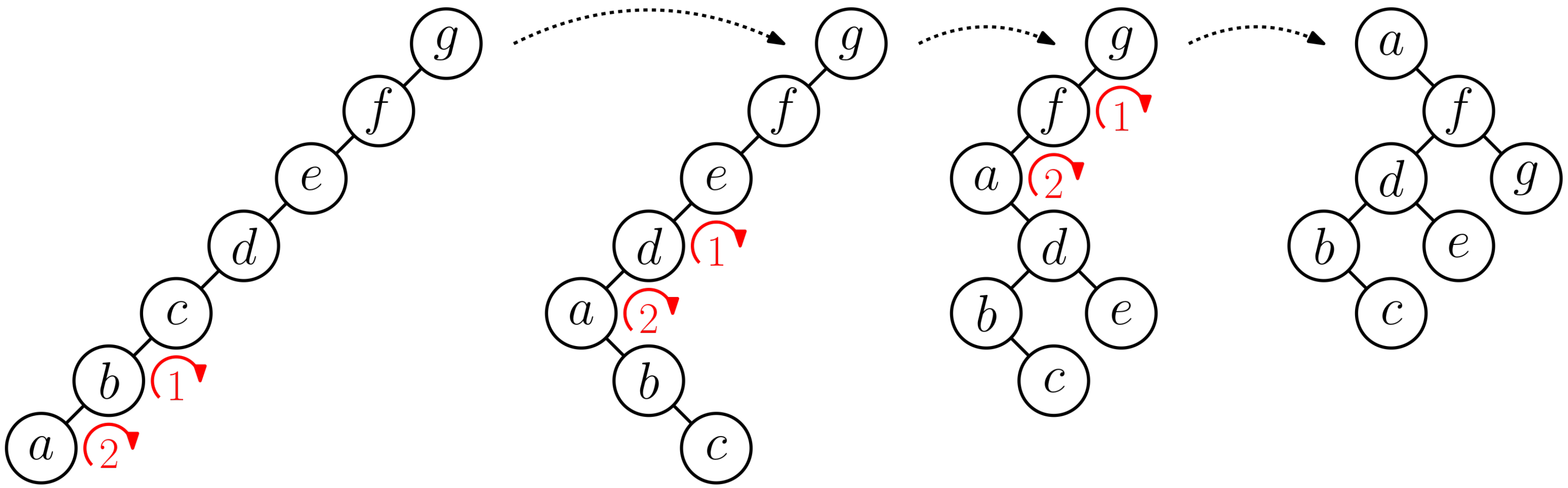
- Here is an idea for a restructuring operation, that doesn't work
  - Let p be the node we wish to access
  - Apply rotations along the path from p back to the root, thus pulling p up to the root
- Unfortunately, while this brings p to the root, the rest of the tree structure may remain poorly balanced



Splay Trees

Fixing our idea

- There is an easy fix, however. Perform rotations two at a time!
- If done properly, the search path length reduces by roughly half

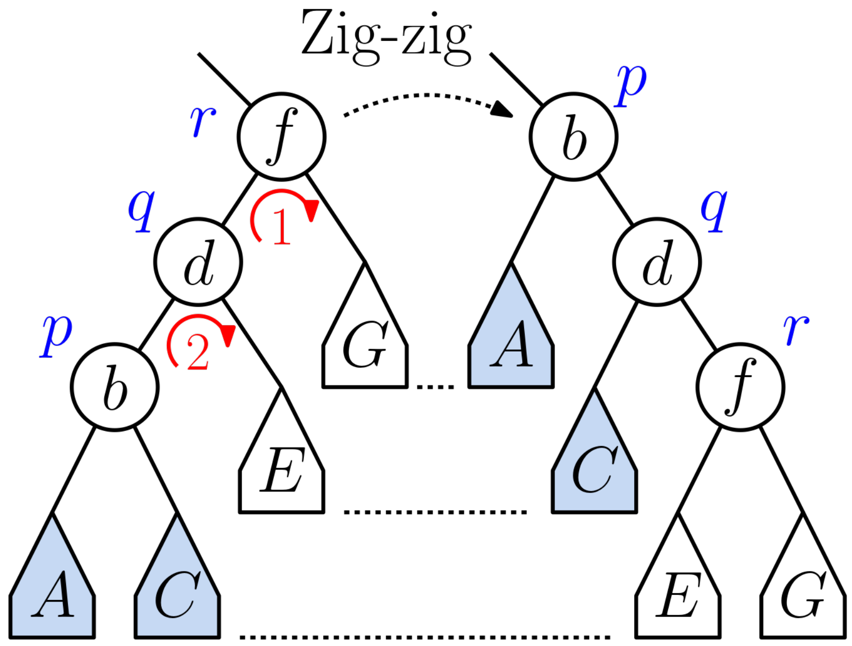


- Can we make this idea rigorous?

Splay Trees

Basic Splay Operations

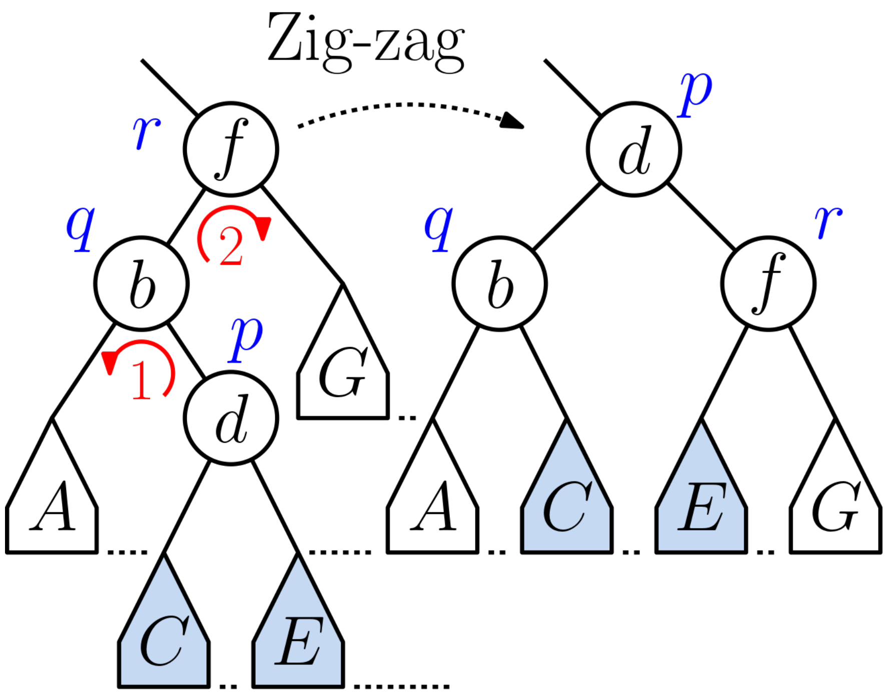
- Let T be a splay tree. The operation T.splay(p) rotates a node p to the root.
- Case 1: (Zig-zig) p is the left-left or right-right grandchild of some node
  - Do two rotations. First at p's grandparent, then at p's parent



Splay Trees

Basic Splay Operations

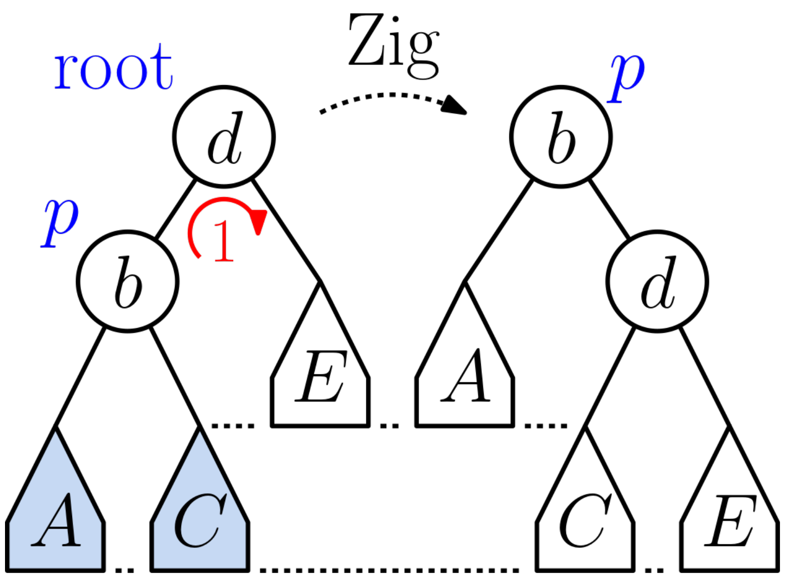
- Case 2: (Zig-zag) p is the left-right or right-left grandchild of some node
  - Do two rotations. First at p's parent, then at p's grandparent



Splay Trees

Basic Splay Operations

- Case 3: (Zig) p is a child of the root
  – Do two a single rotation, pulling p up to the root



- Case 4: (End) p is the root – We're done

Splay Trees

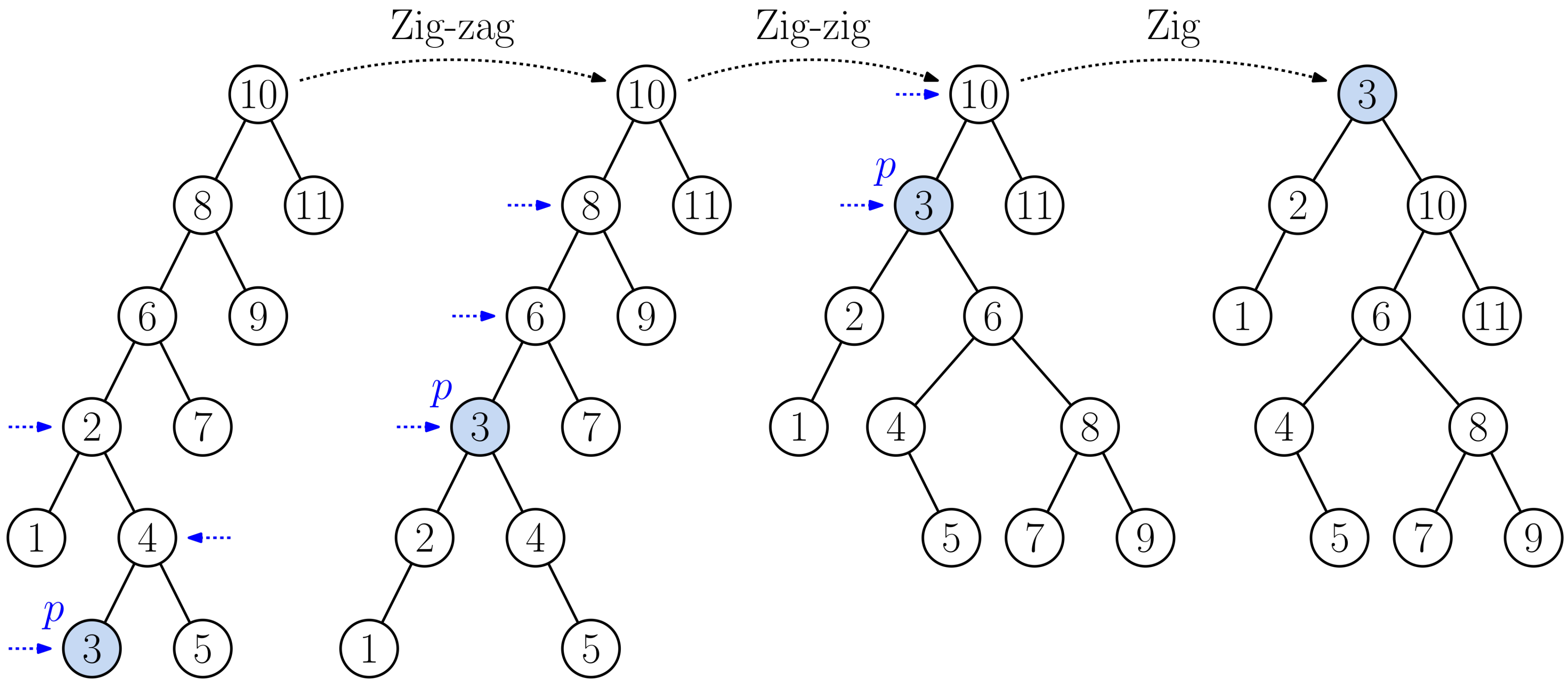
A Self-Adjusting Tree Structure

- `T.splay(x)`:
  - Apply a standard tree descent to find x in the tree.
  - Let p be the node containing x (if present) or the last node visited before falling out (if not). Note that p either contains x or its inorder predecessor or successor
  - Apply zig-zig, zig-zag rotations until almost to root
  - If needed, apply one final zig rotation to finish things off

Splay Trees

A Self-Adjusting Tree Structure

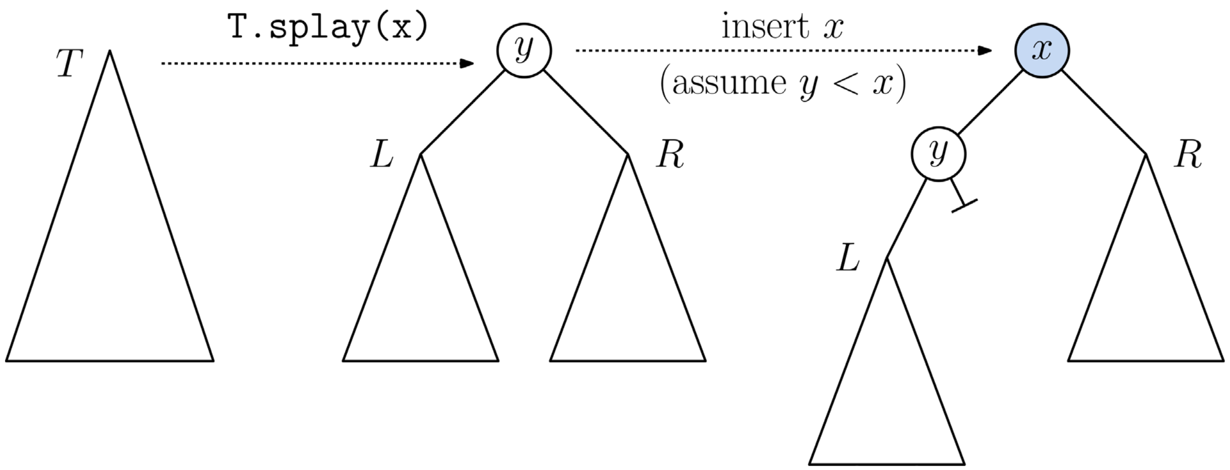
- `T.splay(3)`:



Splay Trees

Dictionary Operations

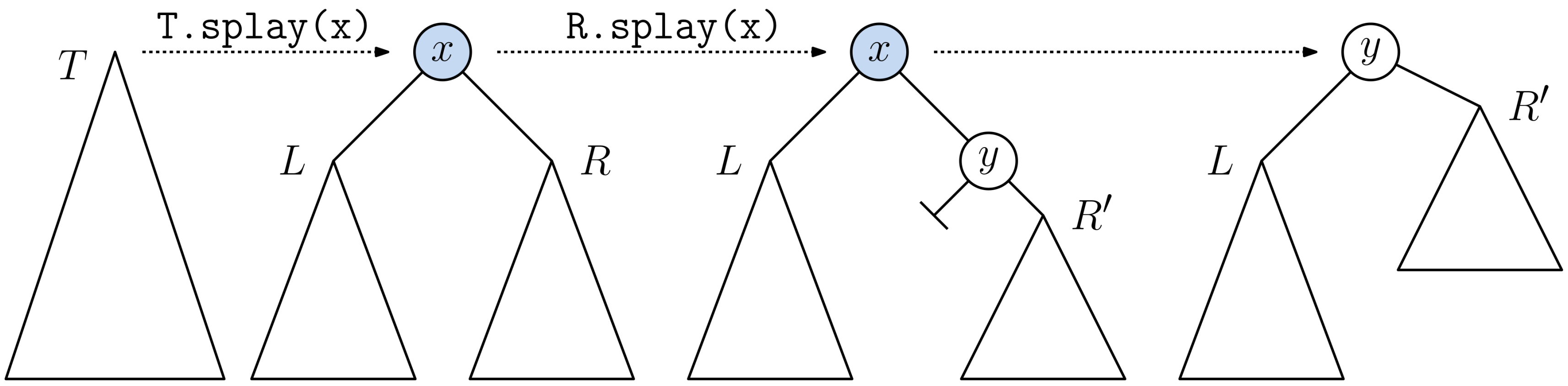
- `T.find(x)`:
  - `T.splay(x)`. Check whether root contains key x
- `T.insert(x, v)`:
  - `T.splay(x)`. If root contains x, duplicate!
  - Let y be root. If $y < x$, link subtrees together as shown below (other case symmetrical)



Splay Trees

Dictionary Operations

- `T.delete(x)`:
  - `T.splay(x)`. Check that x is at root (if not, key not found!)
  - Let L and R be left and right subtrees. If either is `null`, return the other
  - If both are non-null, do `R.splay(x)`
  - New root y is smallest key in R (so its left child is null)
  - Relink trees as shown below



Splay Trees

Amortized Analysis (Optional)

- **Potential**:
  - A function Φ that represents how imbalanced the tree T is
  - Φ is like a bank account that can be spent to balance the tree
  - There must always be sufficient funds in this account
- **Amortized cost**: For any operation, there are two costs to consider:
  - The actual cost of the ith operation (number of rotations): C_i
  - The change in the tree's potential: $\Delta\Phi_i = \Phi_i - \Phi_{i-1}$
  - Amortized cost of ith operation is defined to be: $A_i = C_i + \Delta\Phi_i$
  - Objective: Prove that amortized cost is $O(\log n)$ for every operation
- **Intuition**: Can tolerate a high actual cost, if there is a large decrease in potential

Splay Trees

Amortized Analysis (Optional)

- Potential:
  - For each node p in the tree, $size(p) =$ number of nodes in p's subtree
  - Define $rank(p) = \lg size(p)$ (intuitively, this is ideal height of p's subtree)
  - $\Phi(T) = \sum_{p \in T} rank(p)$
- Rotation Lemma: Given any node p, let $rank(p)$ and $rank'(p)$ be its rank before and after a rotation operation. Then:
  - Amortized cost of zig-zig or zig-zag is $\leq 3(rank'(p) - rank(p))$
  - Amortized cost of zig is $\leq 1 + 3(rank'(p) - rank(p))$
  - (See lecture notes for the proof…it's not easy!)
- Splay Lemma: Amortized cost of `T.splay(p)` is $\leq 1 + 3(rank(root) - rank(p))$
- Corollary: Amortized cost of `T.splay(p)` is $O(\log n)$

Splay Trees

Splay trees have an amazing set of properties

- Consider any sequence S of m accesses to a splay tree of size n
- Balance Theorem: The running time of S is $O(m \log n + n \log n)$
- Static Optimality: Let q_x be the number of times that x is accessed in S. Then the running time of S is $O(m + \sum_i q_x \log {}^{m}/_{q_x})$. This is theoretically optimal (the Entropy of the access distribution)
- Dynamic Finger Theorem: Number the elements 1 through n. Given a sequence of accesses $x_1, \ldots, x_m$, the running time of S is $O(m + \sum_i \log(|x_i - x_{i-1}|+1))$
- Working-Set Theorem: Each time we access x, let $t(x)$ denote the number of accesses since the last time x was accessed, then the running time of S is $O(m + \sum_i \log(t(x) + 1))$
- Scanning Theorem: The time to access all elements in order is $O(n)$

Summary

- Splay Trees
  - Self-adjusting binary search tree
  - Basic operation splay(x) – Brings x to root and reorganizes the tree
    - Zig-zig
    - Zig-zag
    - Zig
  - Splay has the effect of turning long stringing search paths into bushier ones
  - Amortized cost is O(log n) per dictionary operations (find, insert, delete)
  - Splay trees satisfy an impressive set of optimality properties
  - Not widely used, however, because constant factors are high