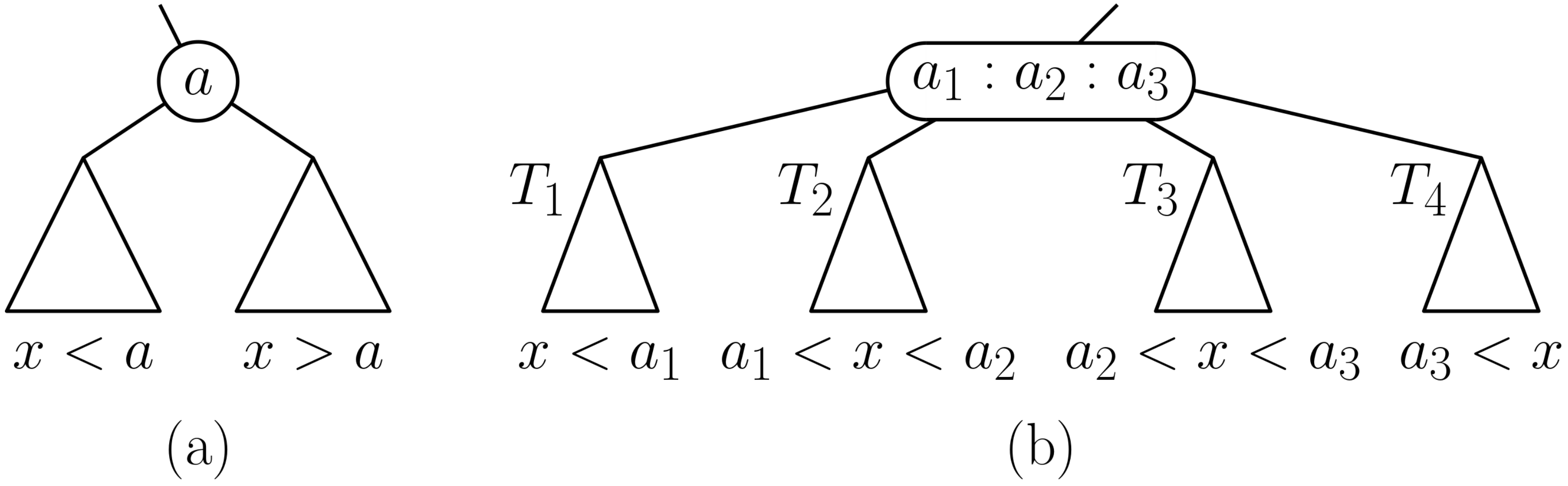
CMSC 420 – 0201 – Fall 2019
Lecture 09

B Trees

B-Trees

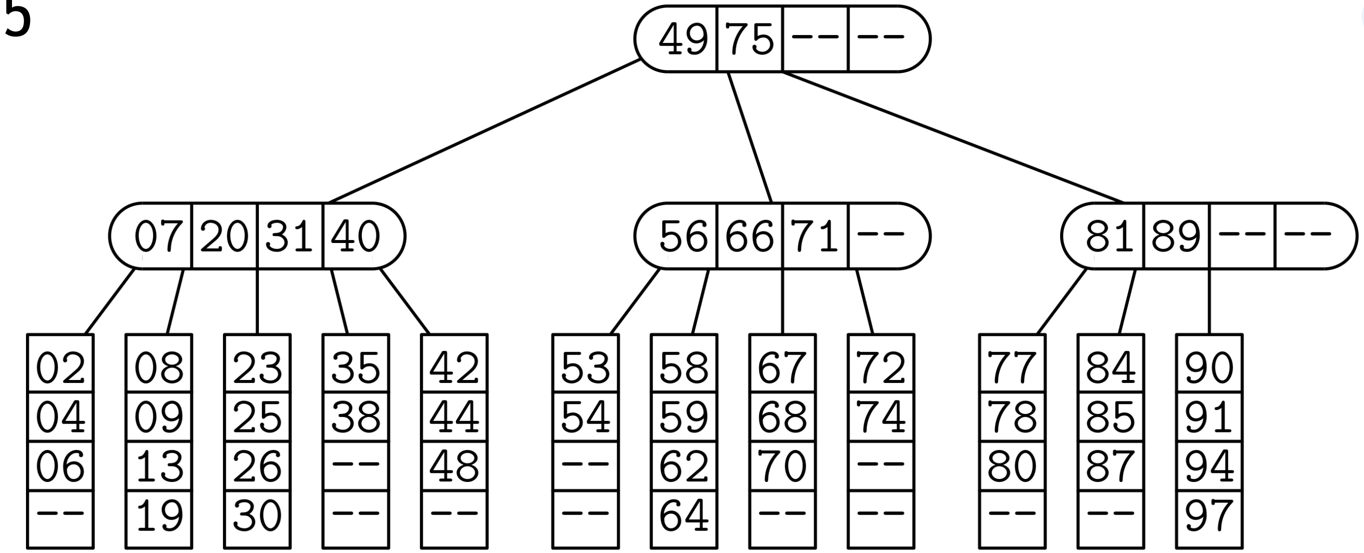
A Search Structure for External Memory

- Binary trees are the method of choice for ordered dictionaries stored in main memory
- On external memory systems (disk), entire blocks (pages) are accessed at once
- We would like each node of our tree to fill a block of external memory
- Multiway search tree – Fan-out depends on block size (e.g., 100)



B-Trees

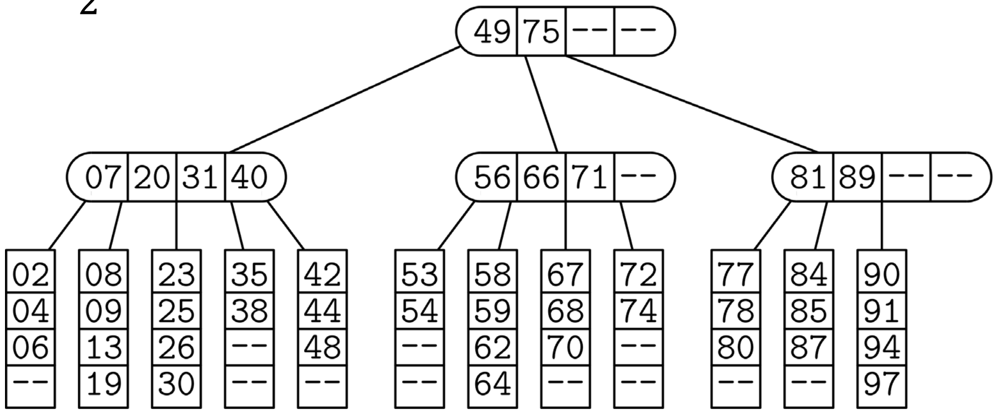
- B-Tree of order m:
  - The root is either a leaf or has between 2 and m children
  - Each non-root node has between $\lceil m/2 \rceil$ to m children (and one fewer keys)
  - All leaves are at the same level of the tree
- Example: B-tree of order 5



B Trees

Height

- **Theorem**: A B-tree of order m with n nodes has height at most $(\lg n)/\gamma$, where $\gamma = \lg\frac{m}{2}$.
- **Proof**:
  - With each level, fan-out is at least $m/2$.
  - Number of nodes in a tree of height h is roughly $n = \left(\frac{m}{2}\right)^h$.
  - Solving for h as function of n, implies $h = \lg n / \lg\frac{m}{2}$
- If $m = 100$, height $\leq \lg n / 5.6$



B-Trees

Node structure

- Because number of keys/children may vary, we allocate the maximum allowed storage for each node:

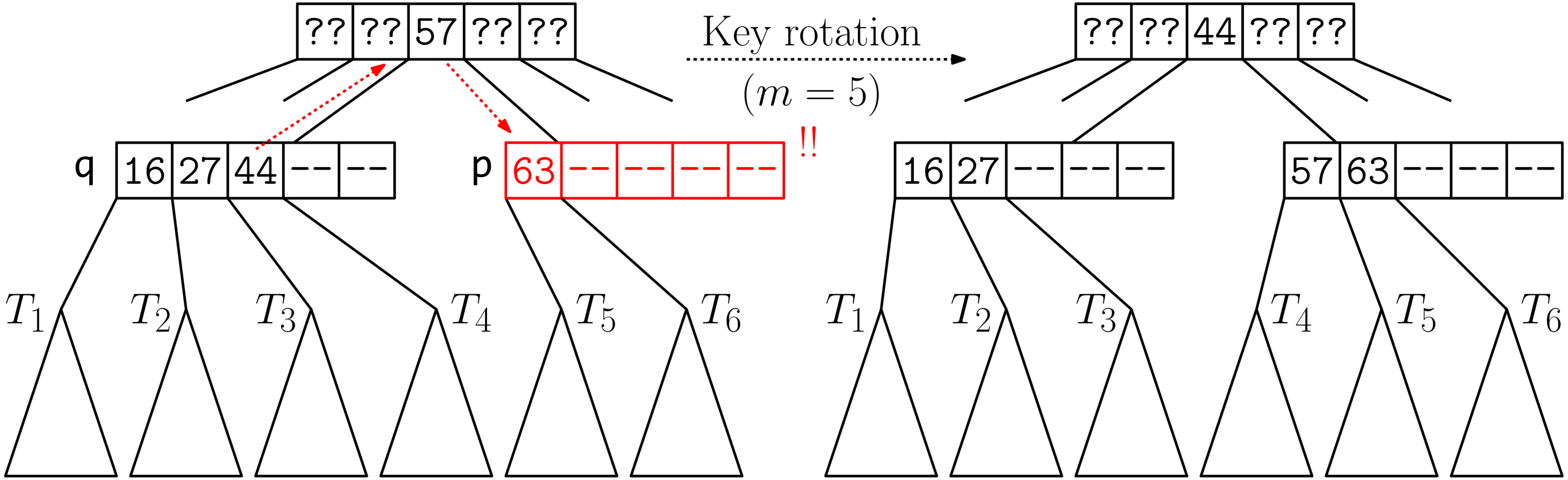
```
const int M = ...               // order of the B-tree

class BTreeNode {
    int         nChildren;      // number of children (from M/2 to M)
    BTreeNode   child[M];       // children pointers
    Key         key[M-1];       // keys
    Value       value[M-1];     // values
}
```

- Setting `M=3` yields a 2-3 tree, `M=4` yields a 2-3-4 tree

B-Trees – Rebalancing operations

Key Rotation (Adoption)

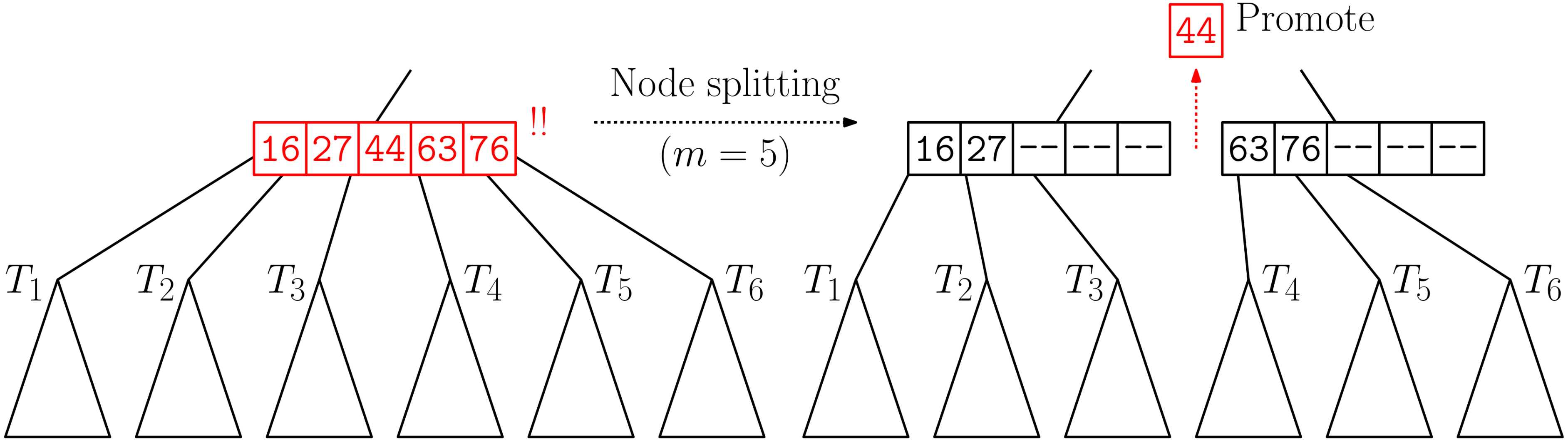
- If node overflows (underflows), and sibling can take (give) a key, rotate the key through the parent out of (into) this node
- Example ($M = 5$): Node p needs a key and sibling q can give one



B-Trees – Rebalancing operations

Node splitting

- If a node has too many children ($m+1$), split the node in half and promote extra key to parent
- New nodes have $m' = \left\lceil \frac{m}{2} \right\rceil$ and $m'' = (m+1) - m'$ children, respectively



B-Trees – Rebalancing operations

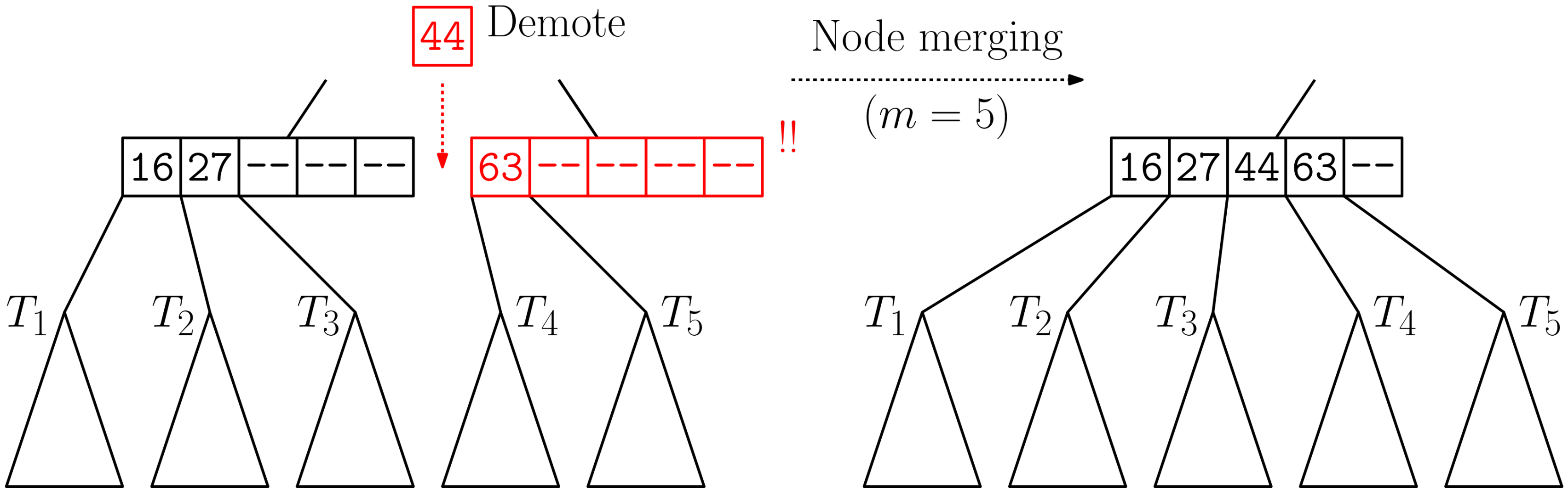
Node splitting

- Need to prove that new node sizes are valid
- **Lemma 1**: For all $m \geq 2$, $\left\lceil \frac{m}{2} \right\rceil \leq m', m'' \leq m$
- **Proof**: This is clearly true for m'. Suffices to consider just m''.
  - Case 1 (m is even):
    - $\implies \left\lceil \frac{m}{2} \right\rceil = \frac{m}{2} \implies m'' = m + 1 - \frac{m}{2} = \frac{m}{2} + 1.$
    - The lemma reduces to proving that $\frac{m}{2} \leq \frac{m}{2} + 1 \leq m$, which is clearly true for any $m \geq 2$.
  - Case 2 (m is odd):
    - $\implies \left\lceil \frac{m}{2} \right\rceil = \frac{m+1}{2} \implies m'' = m + 1 - \frac{m+1}{2} = \frac{m+1}{2}.$
    - The lemma reduces to proving that $\frac{m+1}{2} \leq \frac{m+1}{2} \leq m$, which is clearly true for any $m \geq 1$.

B-Trees – Rebalancing operations

Node merging

- If a node has too few children ($\left\lceil \frac{m}{2} \right\rceil - 1$), and both siblings have the minimum ($\left\lceil \frac{m}{2} \right\rceil$), merge node with sibling and demote one key from parent.
- The new node has size $m''' = \left(\left\lceil \frac{m}{2} \right\rceil - 1\right) + \left\lceil \frac{m}{2} \right\rceil = 2\left\lceil \frac{m}{2} \right\rceil - 1$



B-Trees – Rebalancing operations

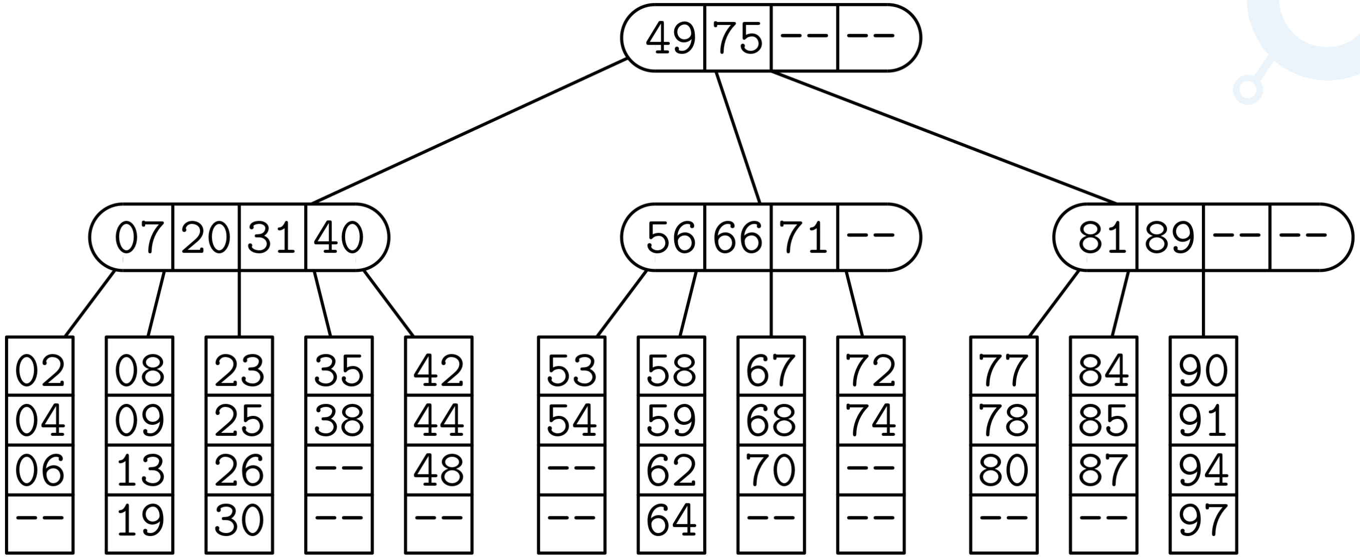
Node merging

- Need to prove that new node size is valid
- Lemma 2: For all $m \geq 2$, $\left\lceil \frac{m}{2} \right\rceil \leq m''' \leq m$
- Proof:
  - Case 1 (m is even):
    - $\Longrightarrow \left\lceil \frac{m}{2} \right\rceil = \frac{m}{2} \Longrightarrow m''' = 2\left(\frac{m}{2}\right) - 1 = m - 1.$
    - The lemma reduces to proving that $\frac{m}{2} \leq m - 1 \leq m$, which is clearly true for any $m \geq 2$.
  - Case 2 (m is odd):
    - $\Longrightarrow \left\lceil \frac{m}{2} \right\rceil = \frac{m+1}{2} \Longrightarrow m''' = 2\left\lceil \frac{m}{2} \right\rceil - 1 = 2\frac{m+1}{2} - 1 = m.$
    - The lemma reduces to proving that $\frac{m+1}{2} \leq m \leq m$, which is clearly true for any $m \geq 1$.

B-Trees – Dictionary operations

Find operation

- `Find(Key x):`
  - Finding a key is analogous to 2-3 trees
  - Descend the tree from the root
  - Let $a_1 < a_2 < \cdots < a_{j-1}$ be keys of current node (convention: $a_0 = -\infty, a_j = +\infty$)
  - Let $T_1, T_2, \ldots, T_j$ be children
  - Find i such that $a_{i-1} < x \leq a_i$:
    - If $a_i = x$, found it
    - Else, if node is leaf, not found
    - Else, search T_i



B-Trees - Dictionary operations

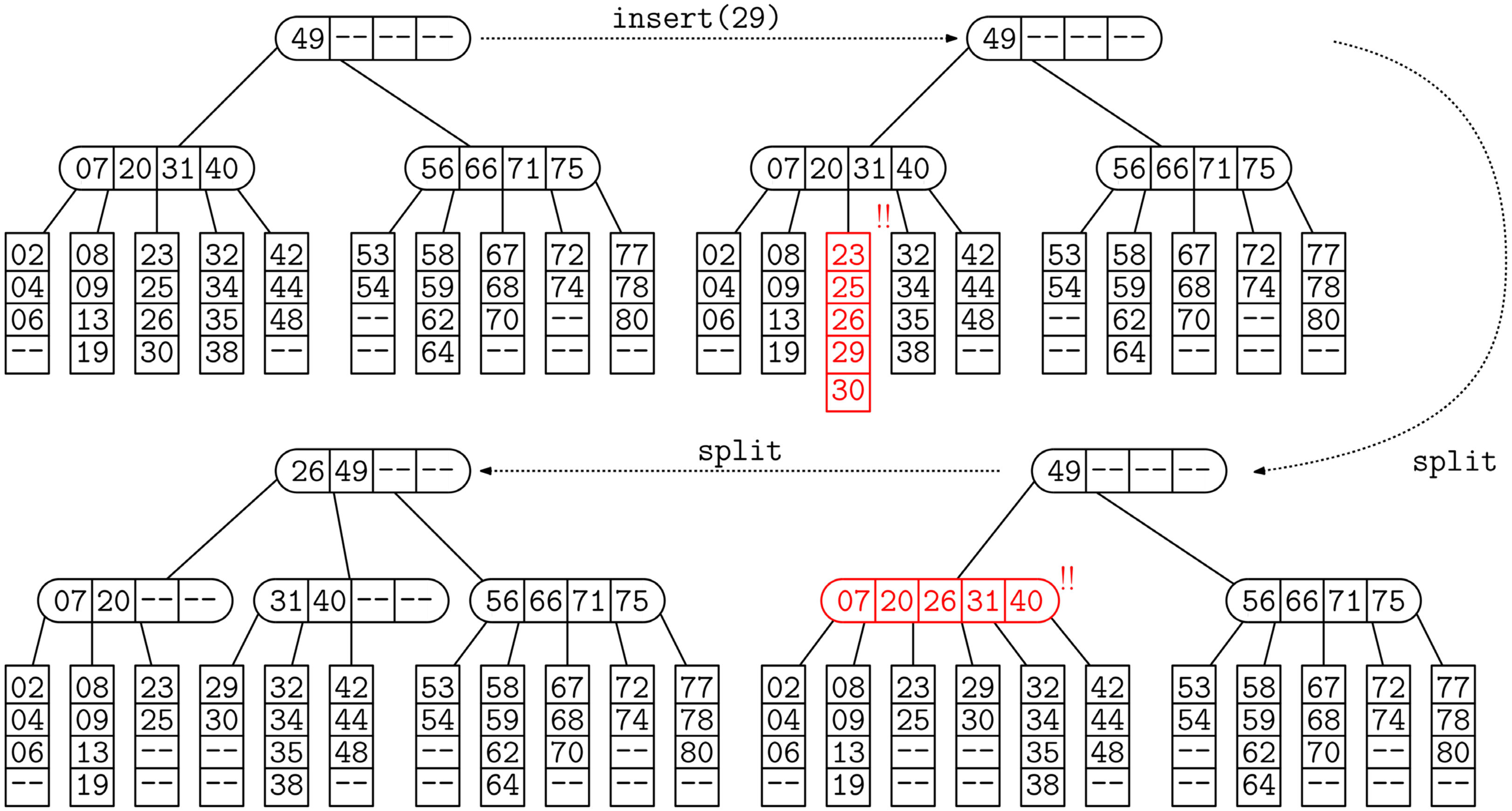
Insertion operation

- `insert(Key x, Value v):`
  - Find the leaf node where x belongs
  - If x is already here - Error
  - Else, insert new `(x,v)` pair in this leaf
  - If node is overfull, attempt key rotation with siblings
  - If siblings are both full, split this node
  - One key is promoted to parent, rebalance the parent recursively
- Note: Key rotation and splitting are both options. Key rotation is preferred because it is less costly and improves space utilization

B-Trees – Dictionary operations

Insertion operation

- Example: `insert(29) (M=5)`



B-Trees - Dictionary operations

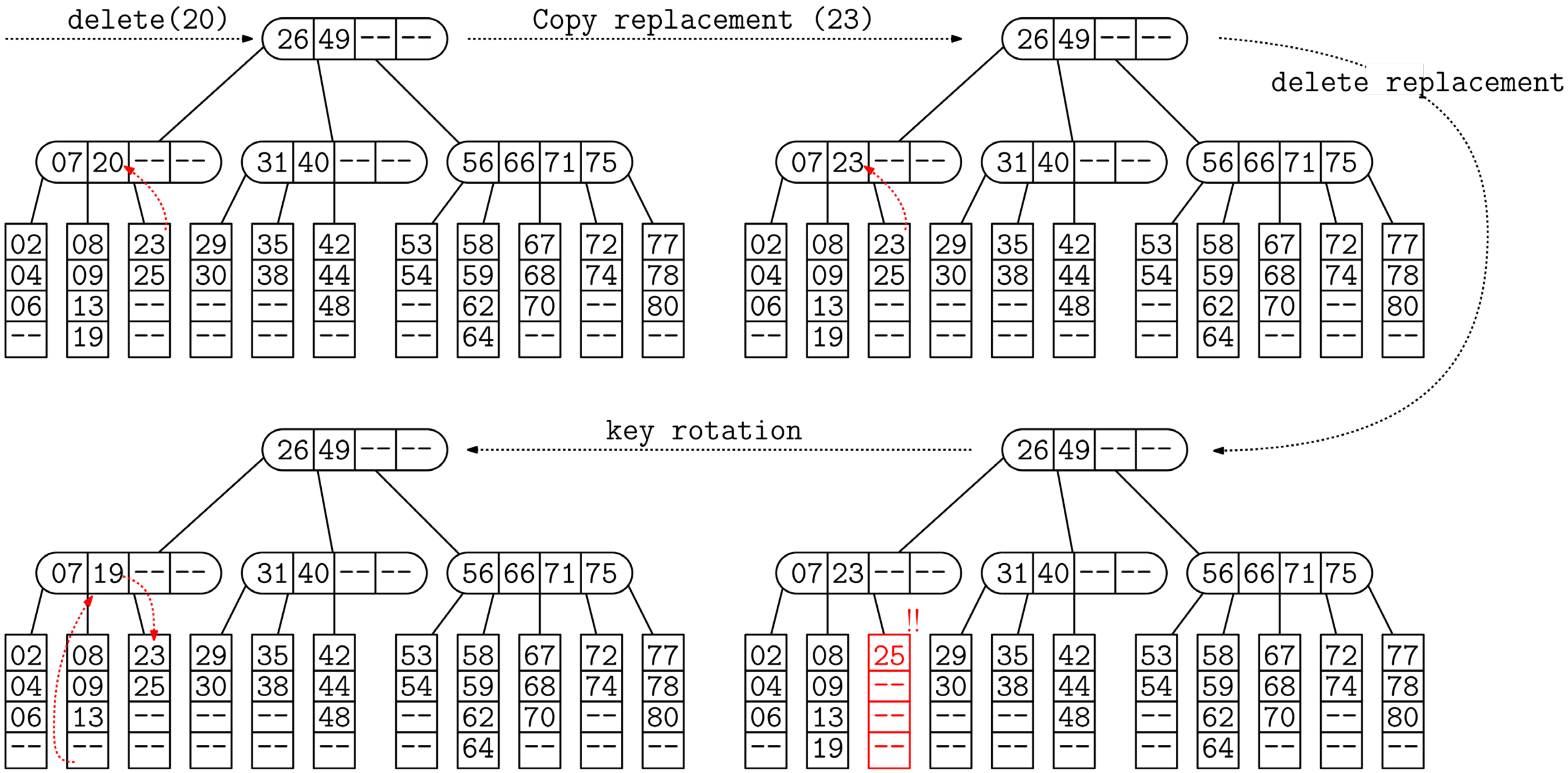
Deletion operation

- `delete(Key x, Value v):`
  - Find the node containing x
  - If not found - Error
  - If not in leaf, find suitable replacement key from leaf level (largest in left subtree or smallest in right subtree), and copy it here
  - Delete the replacement key:
    - If node is underfull, attempt key rotation with siblings
    - If both siblings are minimal, merge this node with either sibling
    - One key is demoted from parent, rebalance the parent recursively

B-Trees - Dictionary operations

Deletion operations

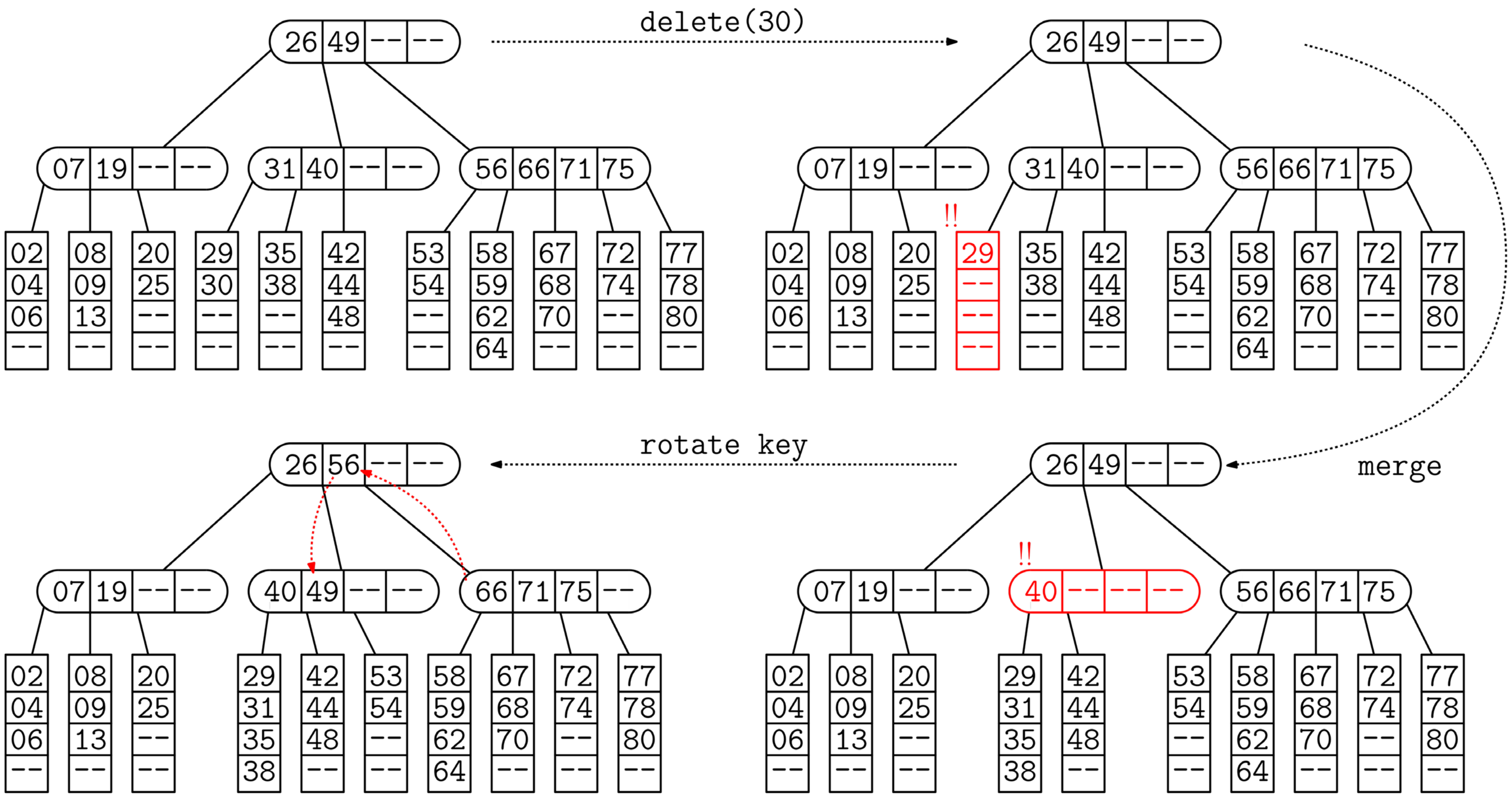
- Example: `delete(20) (M=5)`



B-Trees – Dictionary operations

Deletion operations

- Example: `delete(30) (M=5)`



B-Trees - Variants

B+ Trees

- There are a number of variants of B-trees.
- B+ trees: A popular variant, used in disk storage
  - Key-value pairs are stored only at leaves
  - Internal nodes need only store keys, not values. (Saves space, bigger fan-out implies lower tree heigth, fewer disk accesses)
  - Leaf nodes do not need to waste space for child pointers
  - Each leaf node has a pointer to the next leaf node in the sequence. (Makes it easy to efficiently list all keys in a given range $[x_{min}, x_{max}]$. Find the leaf containing x_{min} and simply keep following next-leaf pointers until coming to x_{max}.

Summary

- B-Trees
  - Multiway search trees – Very popular for disk storage
  - Fan-out m is controllable
  - Height is O(log n / log m)
  - Restructuring generalizes 2-3 tree:
    - Node rotation (adoption)
    - Split
    - Merge
  - Operations (insert, delete, find) run in time O(log n / log m)
- B+ Trees – A practical variant for disk storage