CMSC 420 – 0201 – Fall 2019 Lecture 11

Hashing – Handling Collisions

Hashing - Recap

- We store the n keys in a table containing m entries
- We assume that the table size m is at least a small constant factor larger than n
- We scatter the keys throughout the table using a pseudo-random hash function
  - $h(x) \in [0 \ldots m-1]$
  - Store x at entry $h(x)$ in the table
- Sometimes different keys collide: $x \neq y$, but $h(x) = h(y)$

Hashing - Recap

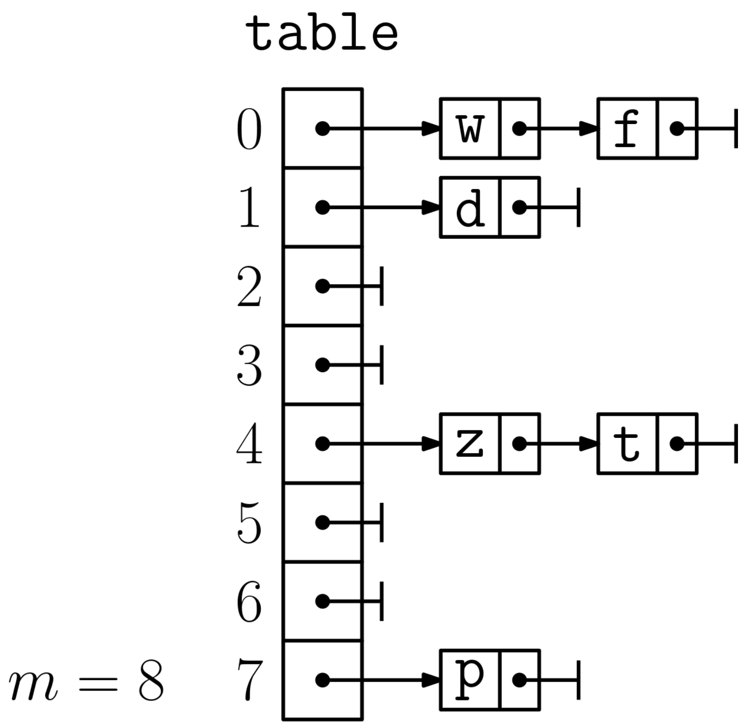
Defining issues

- What is the hash function? Recall common methods:
  - Multiplicative hashing: $h(x) = (ax) \bmod p \bmod m$ (for $a \neq 0$ and prime p)
  - Linear hashing: $h(x) = (ax + b) \bmod p \bmod m$ (for $a \neq 0$ and prime p)
  - Polynomial: $x = (c_0, c_1, c_2, c_3, \ldots,)$, $h(x) = (c_0 + c_1 p + c_2 p^2 + c_3 p^3 + \cdots) \bmod m$
  - Universal hashing: $h_{a,b}(x) = ((ax + b) \bmod p) \bmod m$ (where, a and b are random and p is prime)
- How to resolve collisions? We will consider several methods:
  - Separate chaining
  - Linear probing
  - Quadratic probing
  - Double hashing

Separate Chaining

- Given a hash table `table[]` with m entries
- `table[i]` stores a linked list containing the keys x such that $h(x) = i$

```
insert("d")    h("d") = 1
insert("z")    h("z") = 4
insert("p")    h("p") = 7
insert("w")    h("w") = 0
insert("t")    h("t") = 4
insert("f")    h("f") = 0
```



Separate Chaining

Hash operations reduce to linked-list operations

- `insert(x, v)`: Compute `i=h(x)`, invoke `table[i].insert(x,v)`
- `delete(x)`: Compute `i=h(x)`, invoke `table[i].delete(x)`
- `find(x)`: Compute `i=h(x)`, invoke `table[i].find(x)`

```
insert("d")     h("d") = 1
insert("z")     h("z") = 4
insert("p")     h("p") = 7
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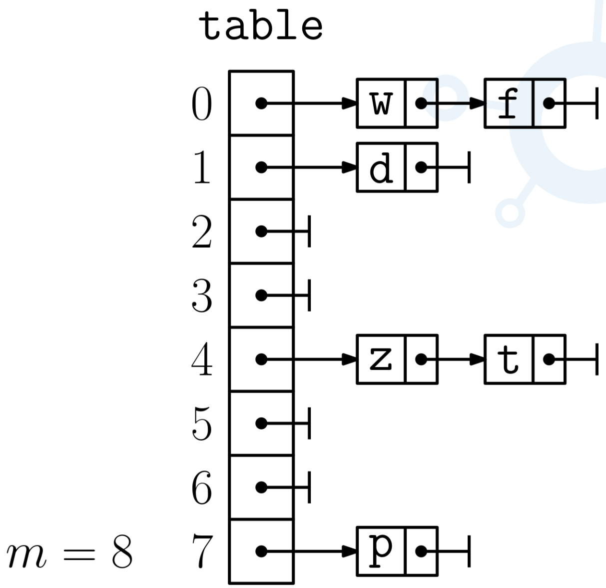


Separate Chaining

Load factor and running time

- Given a hash table `table[m]` containing n entries
- Define load factor: $\lambda = \frac{n}{m}$
- Assuming keys are uniformly distributed, there are on average λ entries per list
- Expected search times:
  - Successful search (key found): Need to search half the list on average

$$S_{SC} = 1 + {}^{\lambda}/_{2}$$

  - Unsuccessful search (key not found): Need to search entire list

$$U_{SC} = 1 + \lambda$$

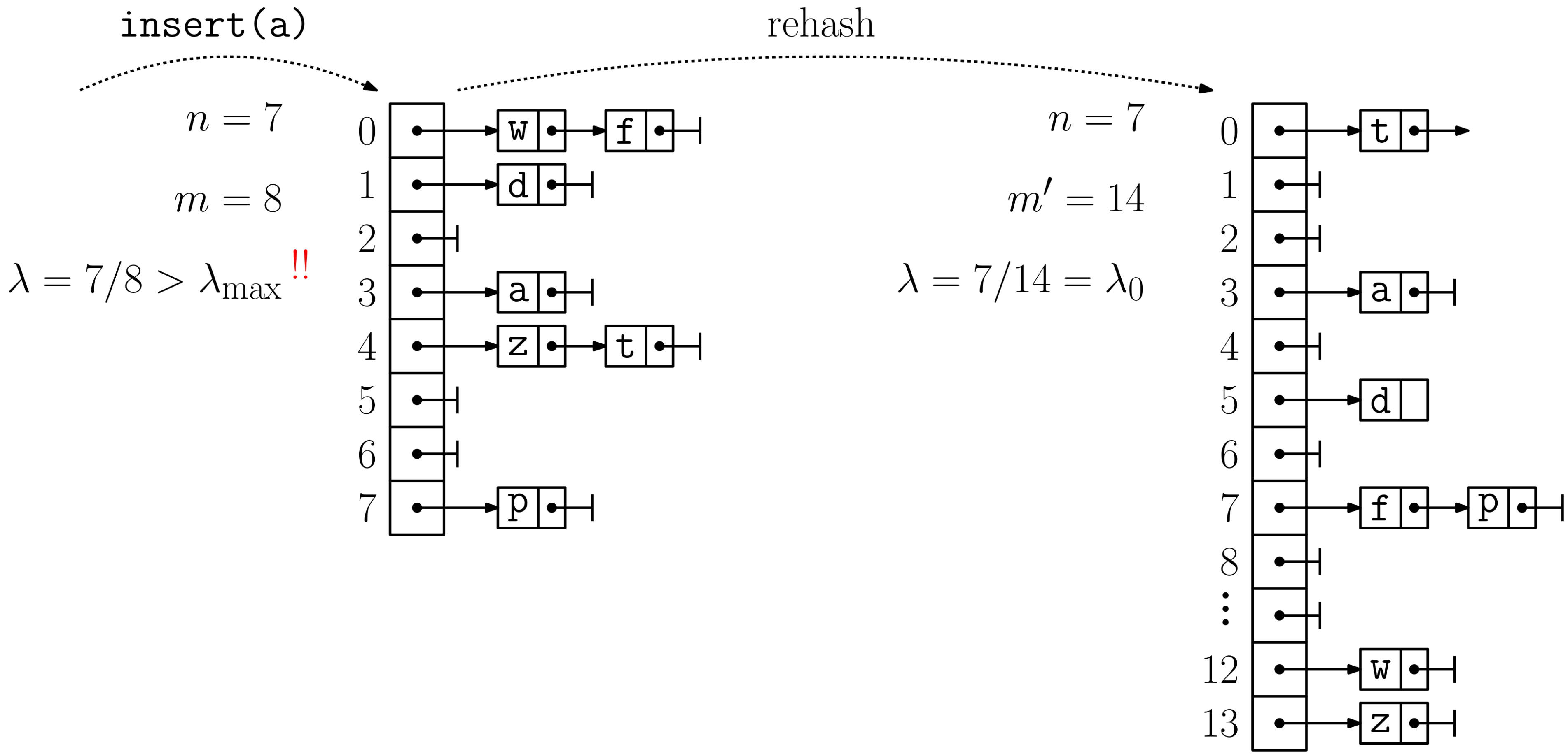
Controlling the Load Factor

Rehashing

- Clearly, we want to keep load factors small, typically $0 < \lambda < 1$
- Select min and max load factors, λ_{min} and λ_{max}, where $0 < \lambda_{min} < \lambda_{min} < 1$
- Define ideal load factor $\lambda_0 = {(\lambda_{min}+\lambda_{max})}/{2}$
- Rehashing (after insertion):
  - If insertion causes load factor to exceed λ_{max}:
    - Allocate a new hash table of size $m' = \frac{n}{\lambda_0}$
    - Create a new hash function h' for this table
    - Rehash all old entries into the new table using h'
  - After rehashing, the load factor is now ${n}/{m'} = \lambda_0$, that is, "ideal"

Rehashing

Example: $\lambda_{min} = \frac{1}{4}$, $\lambda_{max} = \frac{3}{4}$ and $\lambda_0 = \frac{1}{2}$



Controlling the Load Factor

Rehashing

- Underflow: Rehashing can also be applied when the load factor is too small
- Rehashing (after deletion):
  - If deletion causes load factor to be smaller than λ_{min}:
    - Allocate a new hash table of size $m' = \frac{n}{\lambda_0}$
    - Create a new hash function h' for this table
    - Rehash all old entries into the new table using h'
  - After rehashing, the load factor is now $n/m' = \lambda_0$, that is, "ideal"

Rehashing – Amortized Analysis

How expensive is rehashing?

- Rehashing takes time – How bad is it?
- Rehashing takes O(n) time, but once done we are good for a while
- Example:
  - Suppose $m = 1000$, $\lambda_{min} = \frac{1}{4}$ and $\lambda_{max} = \frac{3}{4}$, $\left(\lambda_0 = \frac{1}{2}\right)$
  - After insertion, if $n > \lambda_{max} = 750$, then we allocate a new table of size $m' = n/\lambda_0 \approx 1500$, and rehash the entries here
  - In order to overflow again, we need $n'/m' > \lambda_{max}$
  - That is, we need $n' = 1125$ keys, or equivalently at least $1125 - 750 = 375$ insertions
  - Amortization: We charge the (expensive) work of rehashing to these (cheap) insertions

Rehashing – Amortized Analysis

How expensive is rehashing?

- Theorem: Assuming that individual hashing operations take $O(1)$ time each, if we start with an empty hash table, the amortized complexity of hashing using the above rehashing method with load factors of λ_{min} and λ_{max}, respectively, is at most $1 + 2\lambda_{max}/(\lambda_{max} - \lambda_{min})$
- Proof:
  – Token-based argument: Each time we perform a hash-table operation, we assess 1 unit for the actual operation and save $2\lambda_{max}/(\lambda_{max} - \lambda_{min})$ work tokens for future use
  – Two cases: Overflow and underflow

Rehashing – Amortized Analysis

How expensive is rehashing?

- Token-based argument: Each time we perform a hash-table operation, we assess 1 unit for the actual operation and save $2\lambda_{max}/(\lambda_{max} - \lambda_{min})$ work tokens for future use
- Overflow:
  - Current table has $n \approx \lambda_{max} m$ entries. This is the cost of rehashing
  - Just after the previous rehash, table contained $n' = \lambda_0 m$ entries
  - Since then, we performed at least $n - n' = (\lambda_{max} - \lambda_0)m$ insertions
  - By simple math, we have $\lambda_{max} - \lambda_0 = \lambda_{max} - \frac{\lambda_{max}+\lambda_{min}}{2} = (\lambda_{max} - \lambda_{min})/2$
  - Thus, the number of tokens collected is at least
    $$(2\lambda_{max}/(\lambda_{max} - \lambda_{min})) \cdot (\lambda_{max} - \lambda_0)m = \lambda_{max} m \approx n$$
  - In summary, we have enough tokens to pay for rehashing!
- Underflow: (Similar…see lecture notes)

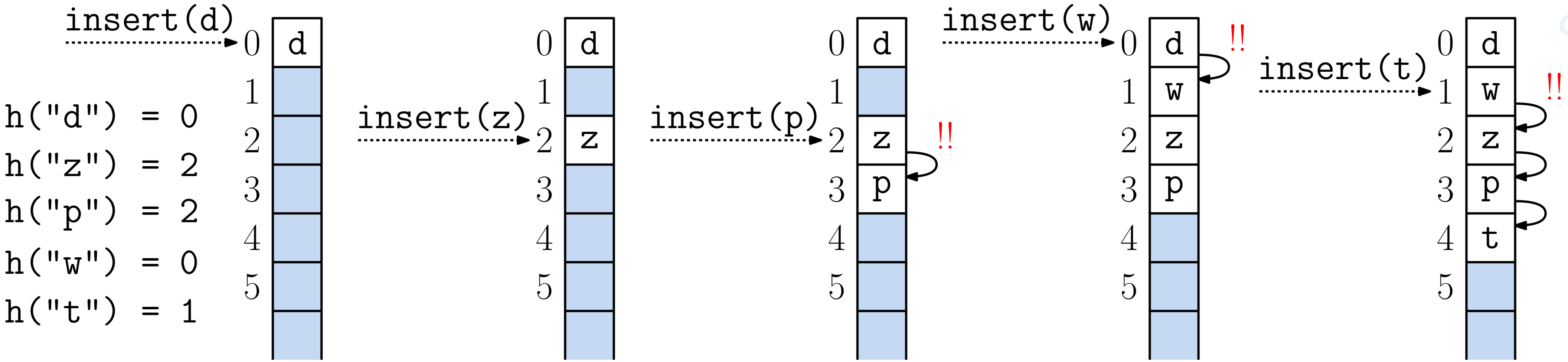
Open Addressing

- Separate chaining requires additional storage. Can we avoid this?
- Store everything in the table
- Requires that $n \leq m$, that is, $\lambda \leq 1$.
- Open Addressing:
  - Special entry "`empty`" indicates that this table entry is unused
  - To insert `x`, first check `table[h(x)]`. If empty, then store here
  - Otherwise, probe subsequent table entries until finding an empty location
  - Which entries to probe? Does it matter?
  - Yes! As the load factor approaches 1, some probe methods have good performance and others do not

Open Addressing – Linear Probing

Quick and dirty (maybe too quick and dirty)

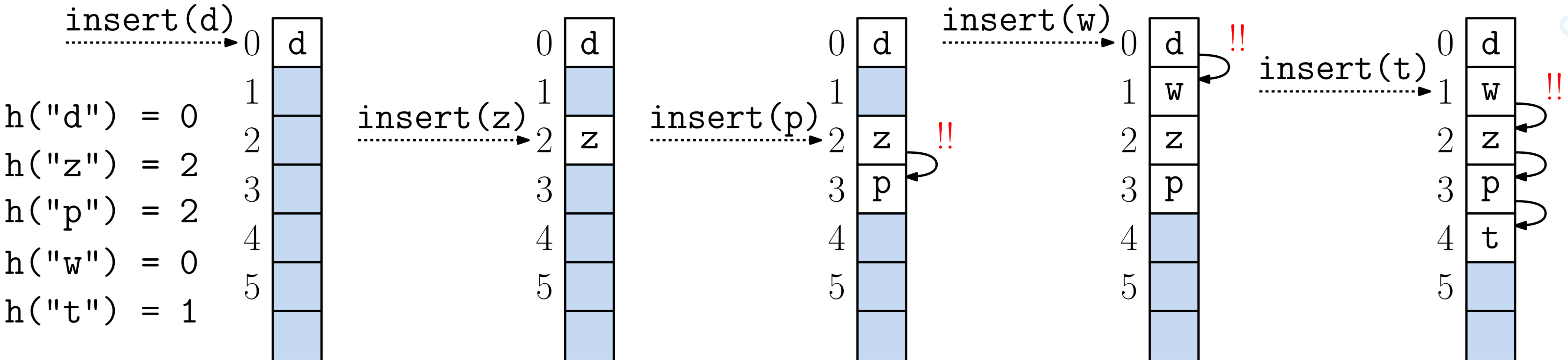
- Linear probing:
  - If `table[h(x)]` is not empty, try `h(x)+1`, `h(x)+2`, …, `h(x)+j`, until finding the first empty entry
  - Wrap around if needed: `table[(h(x)+j) % m]`
- Example:



Open Addressing – Linear Probing

Secondary clustering

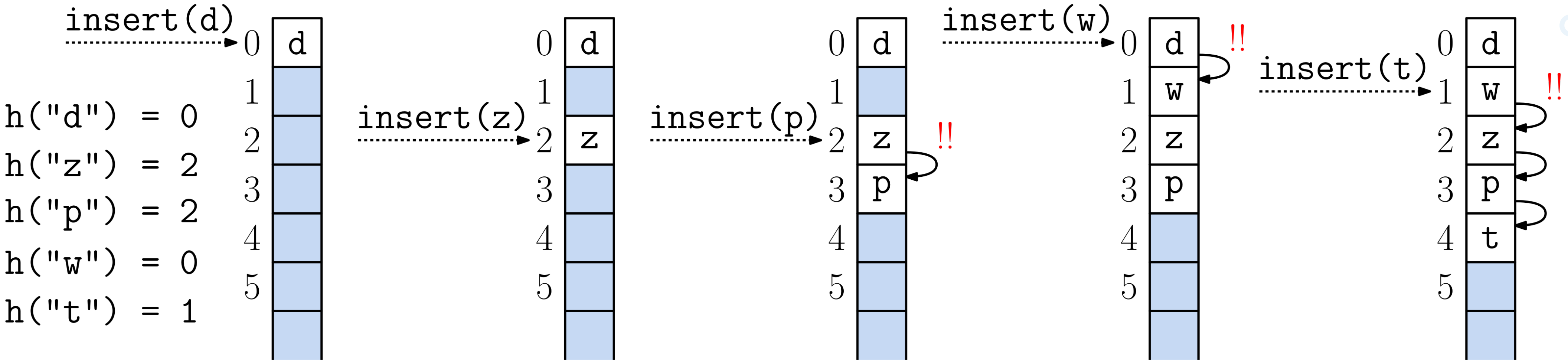
- Primary clustering: Clusters that occurs due to many keys hashing to the same location. (Should not occur if you use a good hash function)
- Secondary clustering: Clustering that occurs because collision resolution fails to disperse keys effectively
- Bad news: Linear probing is highly susceptible to secondary clustering



Open Addressing – Linear Probing

Secondary clustering

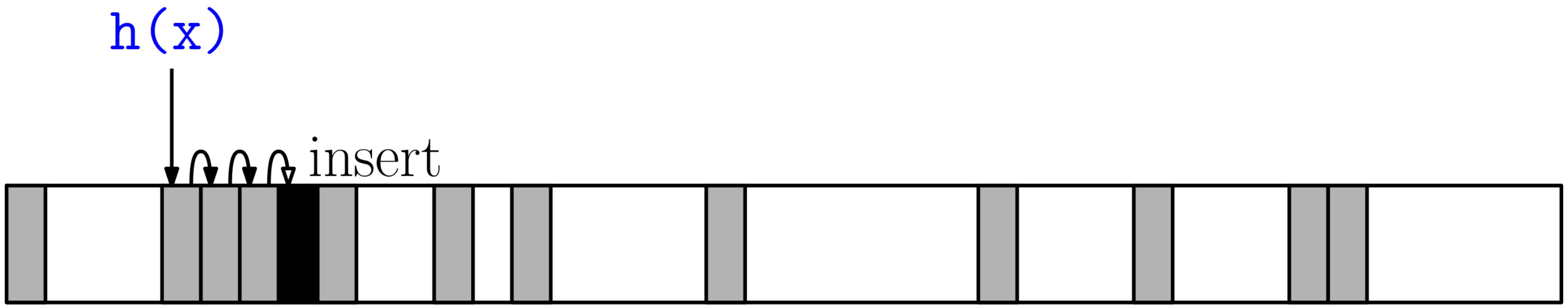
- Expected search times:
  - Successful search (key found): $S_{LP} = \frac{1}{2}\left(1 + \frac{1}{1-\lambda}\right)$
  - Unsuccessful search (key not found): $U_{LP} = \frac{1}{2}\left(1 + \frac{1}{1-\lambda}\right)^2$
  - A table becomes full, $\lambda \to 1$, U_{LP} grows very rapidly



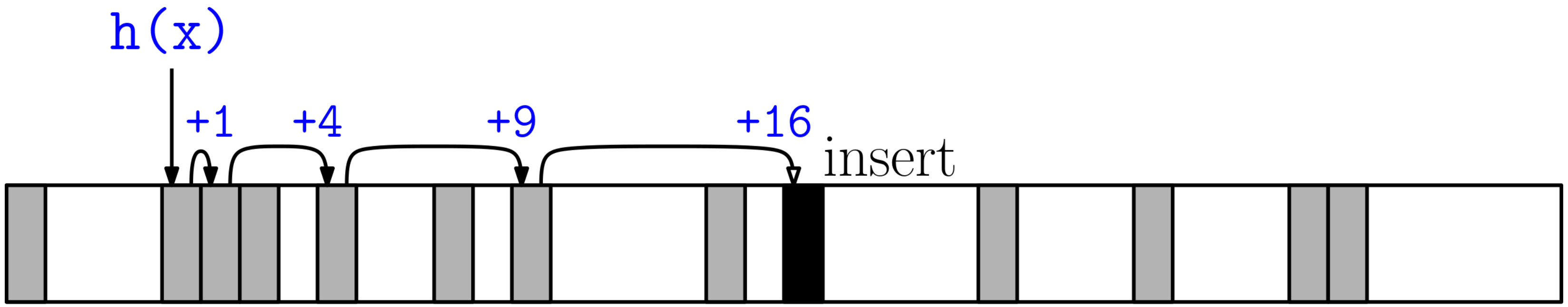
Open Addressing – Quadratic Probing

An attempt to avoid secondary clustering

- Linear probing: $h(x) + 1, 2, 3, \dots, i$ clusters keys very close to the insertion point



- Quadratic probing: $h(x) + 1,4,9, \dots, i^2$ disperses keys better, reducing clustering



Open Addressing – Quadratic Probing

An attempt to avoid secondary clustering

- Quadratic probing: $h(x) + 1, 4, 9, \ldots, i^2$ disperses keys better, reducing clustering
- Let `table[i].key` and `table[i].value` be the key and value
- Cute trick: $i^2 = (i-1)^2 + (2i-1)$. For next offset, add $2i+1$ to previous offset
- Pseudo-code for `find(x)`:

```
Value find(Key x) {
    int c = h(x)                          // initial probe location
    int i = 0                             // probe offset
    while (table[c].key != empty) && (table[c].key != x) {
        c += 2*(++i) - 1                  // next position
        c = c % m                         // wrap around
    }
    return table[c].value                 // return associated value (or null if empty)
}
```

Open Addressing – Quadratic Probing

An attempt to avoid secondary clustering

- Quadratic probing:
  - More formally, the probe sequence is $h(x) + f(i)$, where $f(i) = i^2$
- Complete coverage?
  - Does the probe sequence hit every possible table location?
  - No! For example, if $m = 4$, $i^2 \bmod 4$ is either 0 or 1, never 2 or 3. (Try it!)
- Any hope? Can we select m so that quadratic probing hits all entries?
  - If m is prime of the form $4k + 3$, quadratic probing will hit every table entry before repeating (source: Wikipedia - Related to quadratic residues)
  - If m is a power of 2, and we use $f(i) = \frac{1}{2}(i^2 + i)$, quadratic probing will hit every table entry before repeating (source: Wikipedia)

Open Addressing – Quadratic Probing

An attempt to avoid secondary clustering

- Theorem: If quadratic probing is used, and the table size m is a prime number, the first $\left\lfloor \frac{m}{2} \right\rfloor$ probe sequences are distinct.
- Proof:
  - By contradiction. Suppose that there exist i, j, such that $0 \leq i < j \leq \left\lfloor \frac{m}{2} \right\rfloor$ and $h(x) + i^2$ and $h(x) + j^2$ are equivalent modulo m.
  - Then the following equivalences hold mod m:

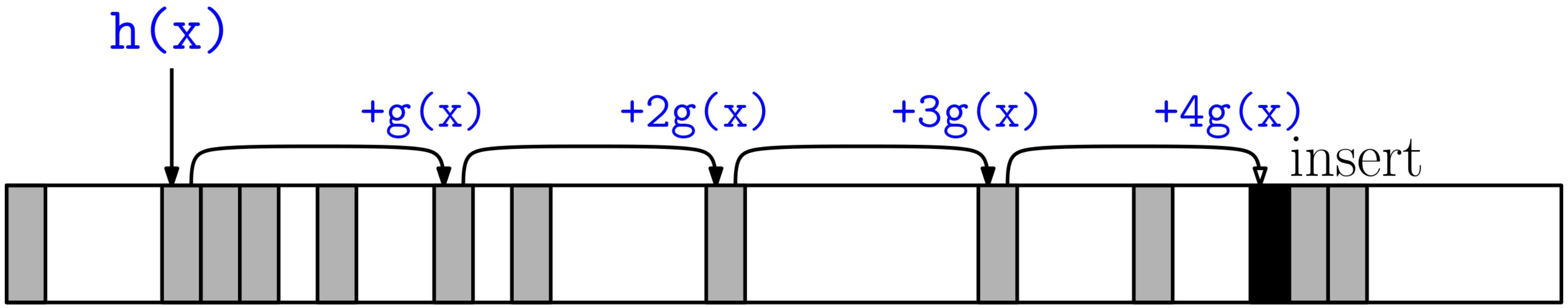
$$i^2 \equiv j^2 \iff i^2 - j^2 \equiv 0 \iff (i+j)(i-j) \equiv 0 \pmod{m}$$

  - Since m is prime, both $i + j$ and $i - j$ must be multiples of m. But since $0 \leq i < j \leq \left\lfloor \frac{m}{2} \right\rfloor$, both quantities are smaller than m, and hence cannot be multiples. Contradiction!

Open Addressing – Double Hashing

Saved the best for last

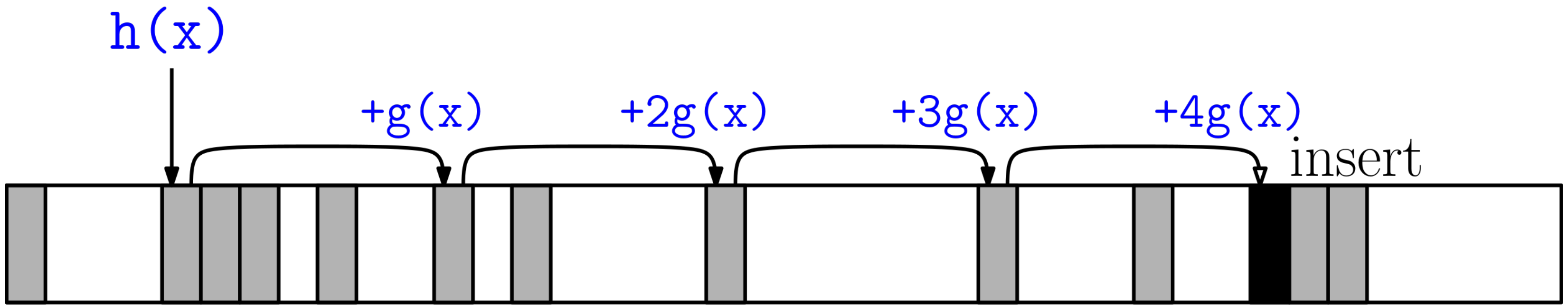
- Linear probing suffers from secondary clustering
- Quadratic probing may fail to hit all cells
- Double hashing:
  - Probe offset is based on a second hash function $g(x)$
  - Probe sequence: $h(x)$, $h(x) + g(x)$, $h(x) + 2g(x)$, $h(x) + 3g(x)$, …



Open Addressing – Double Hashing

Saved the best for last

- Double hashing:
  - Probe offset is based on a second hash function $g(x)$
  - Probe sequence: $h(x)$, $h(x) + g(x)$, $h(x) + 2g(x)$, $h(x) + 3g(x)$, …
- Will this hit all entries before cycling?
  - Yes! If m and $g(x)$ are relatively prime, share no common factors. (E.g., Making $g(x)$ a prime greater than m guarantees this)



Open Addressing – Double Hashing

Saved the best for last

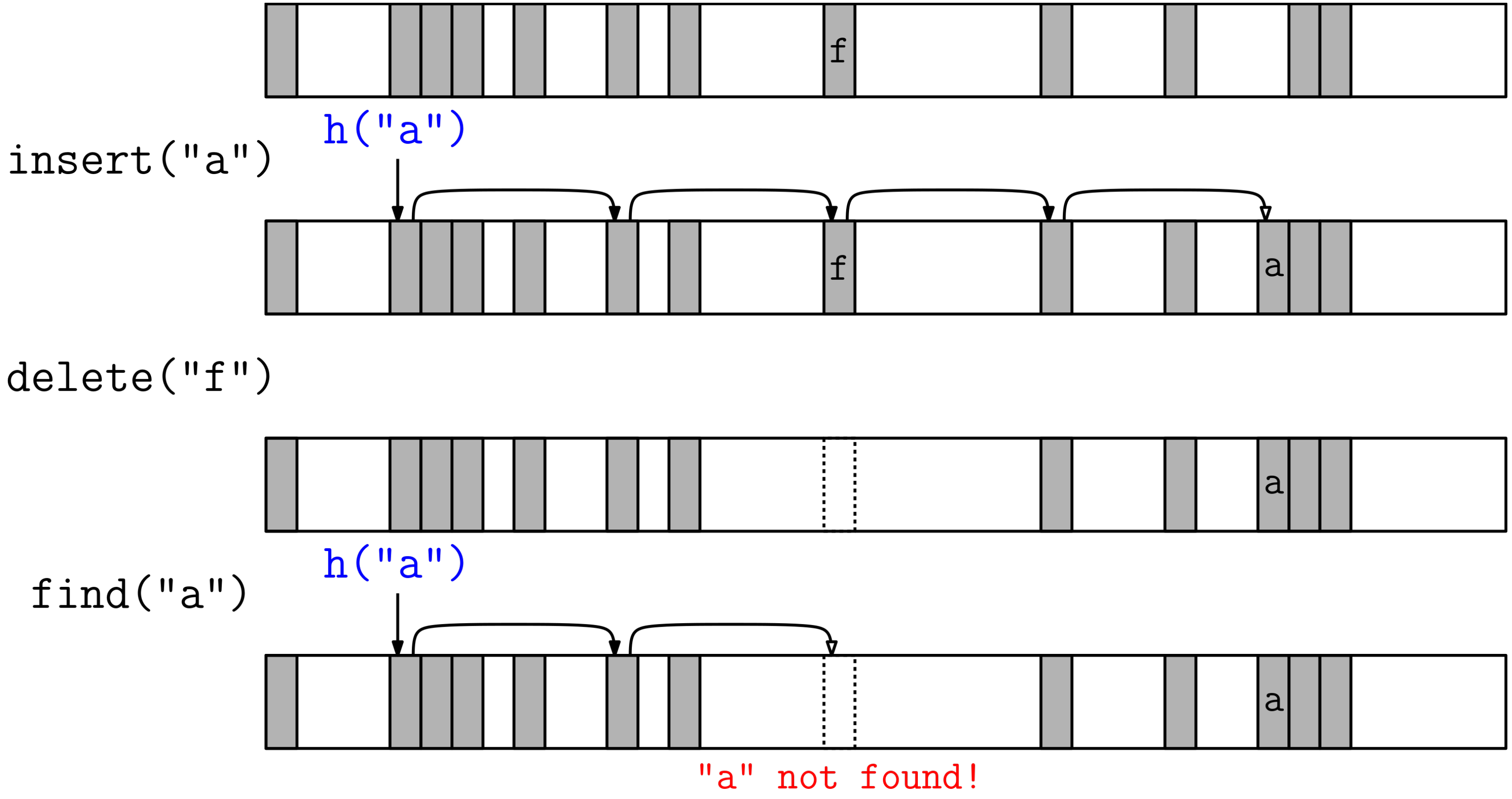
- Double hashing has the best search times among all the methods covered so far:
  - Successful search (key found): $S_{DH} = \frac{1}{\lambda} \ln\left(\frac{1}{1-\lambda}\right)$
  - Unsuccessful search (key not found): $U_{DH} = \frac{1}{1-\lambda}$
- Some sample values:

λ	0.5	0.75	0.9	0.95	0.99
$U(\lambda)$	2.00	4.00	10.0	20.0	100
$S(\lambda)$	1.39	1.89	2.56	3.15	4.65

Open Addressing – Deletion

Deletion requires care!

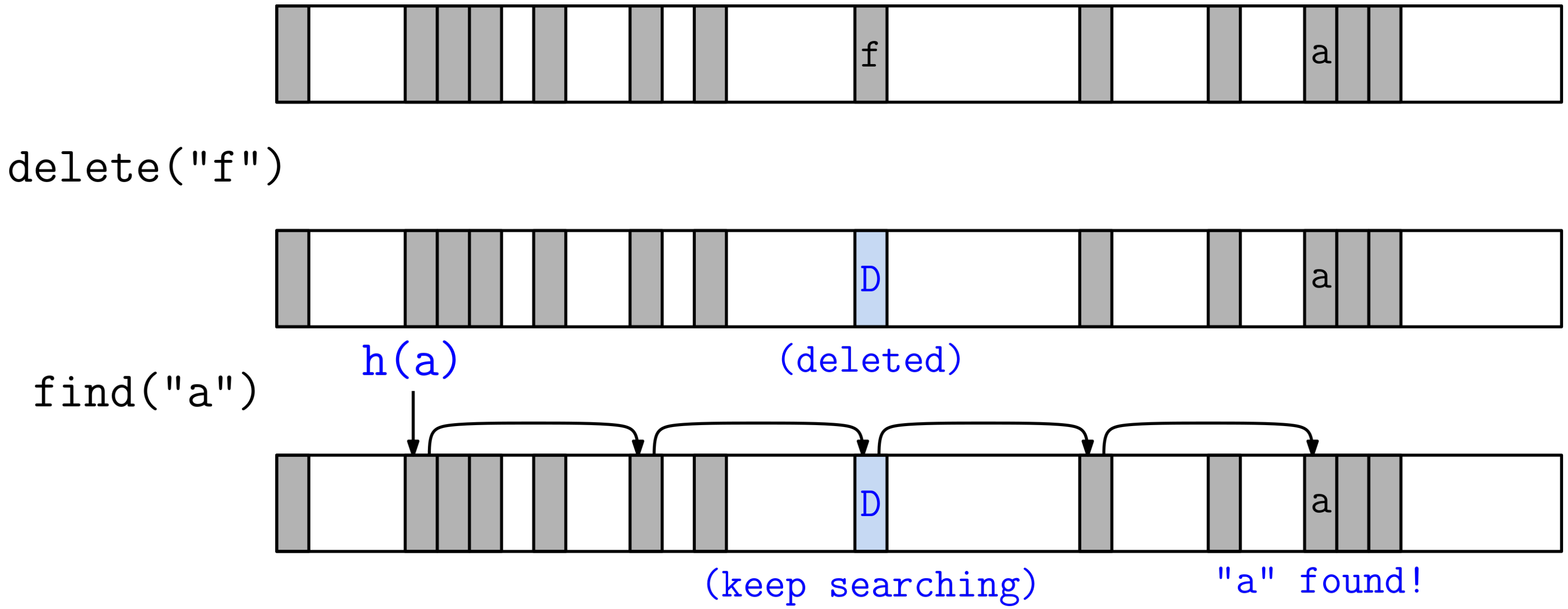
- Deleted entries can create the illusion we are at the end of the probe sequence



Open Addressing - Deletion

Quick and dirty fix

- Special entry "`deleted`": The item at this location has been deleted
  - When searching: don't stop here
  - When inserting: a key can be placed here



Hashing – Further Refinements

- Hashing has been around a long time, and numerous refinements have been proposed
- Example: Brent's Method
  - When using double hashing, multiple probe sequences (with different values of g(x)) may overlap at a common cell of the hash table, say `table[i]`
  - One of these sequence places its key in `table[i]`, and for the other, this wasted cell just adds to the search times
  - To improve average search times, we should give ownership of the cell to the longer of the two probe sequences (and move the other key later in its probe sequence)
  - Brent's algorithm optimizes the placement of keys in overlapping probe sequences

Summary

- Hashing – The fastest implementation of the dictionary data type
  - Does not support ordered operations (min, max, range query, kth smallest, …)
  - Key elements:
    - Hash function - Linear, Polynomial, Universal hashing
    - Collision resolution
      - Separate chaining
      - Open Addressing:
        - Linear probing
        - Quadratic probing
        - Double hashing
  - Analysis: Load factors, rehashing and amortized efficiency