CMSC 420 – 0201 – Fall 2019
Lecture 12

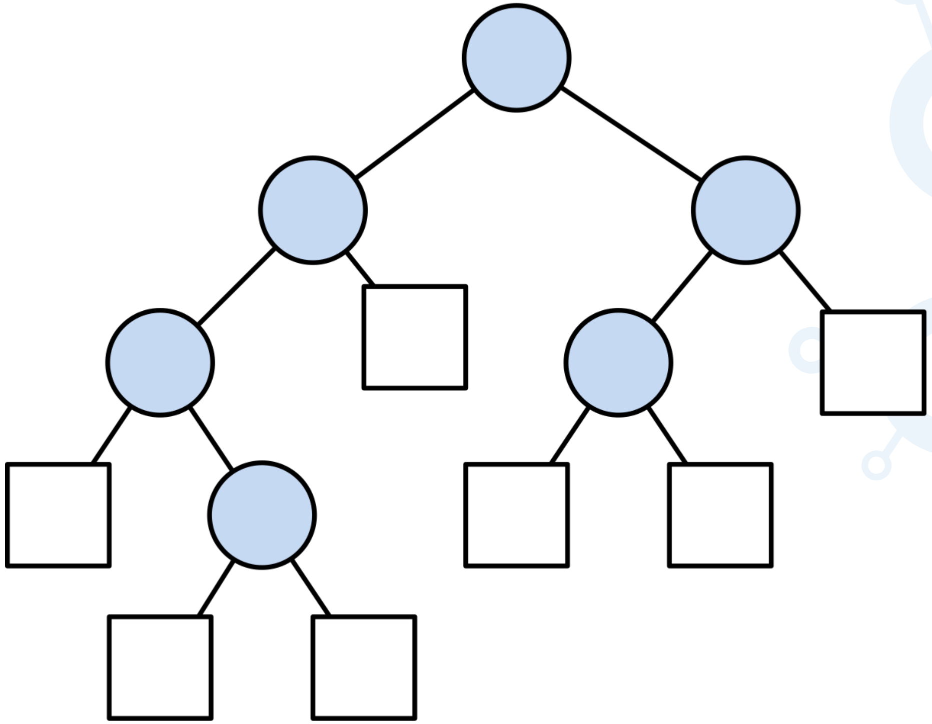
Extended and Scapegoat Trees

Overview

- In today's lecture, we will discuss two unrelated topics that arise in the programming assignment:
  - Extended Binary Search Trees
  - Scapegoat Trees
- We will also discuss the SG Tree, which is featured in the Programming Project, Part 1

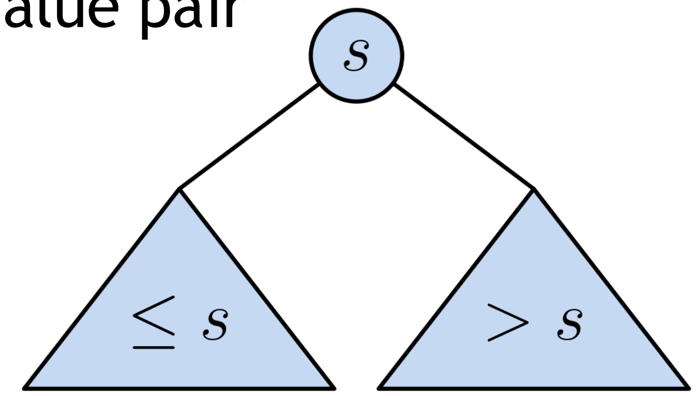
Extended Binary Search Trees

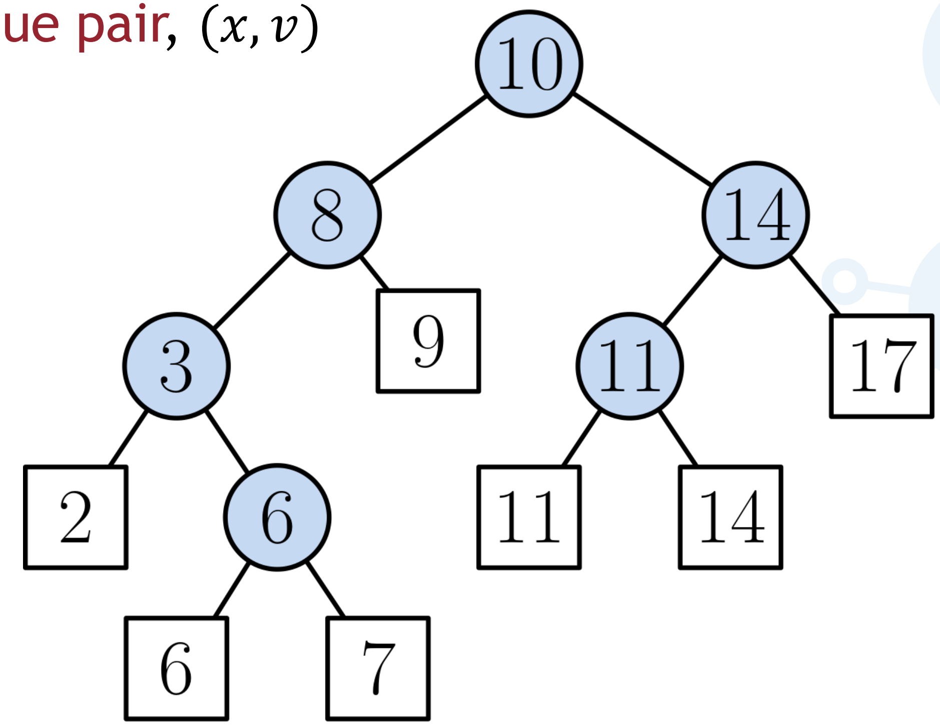
- Extended Binary Tree (from Lecture 3)
  - Internal nodes: Have exactly 2 children
  - External nodes: Have 0 children
- Basic properties
  - Any extended binary tree with n internal nodes has $n + 1$ leaves



Extended Binary Search Trees

- Extended Binary Search Trees
  - Each external node contains an entry, a key-value pair, (x, v)
  - Each internal node contains a splitter, s
    - If $x \leq s$ $\rightarrow$ Left subtree
    - If $x > s$ $\rightarrow$ Right subtree
- Note that a key can be both a splitter and part of a key-value pair



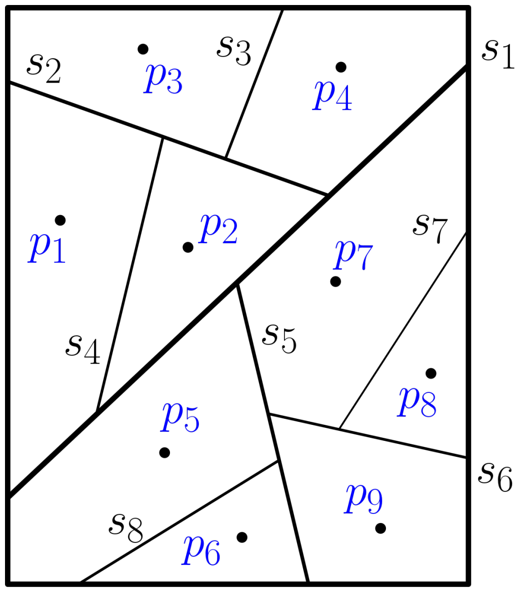


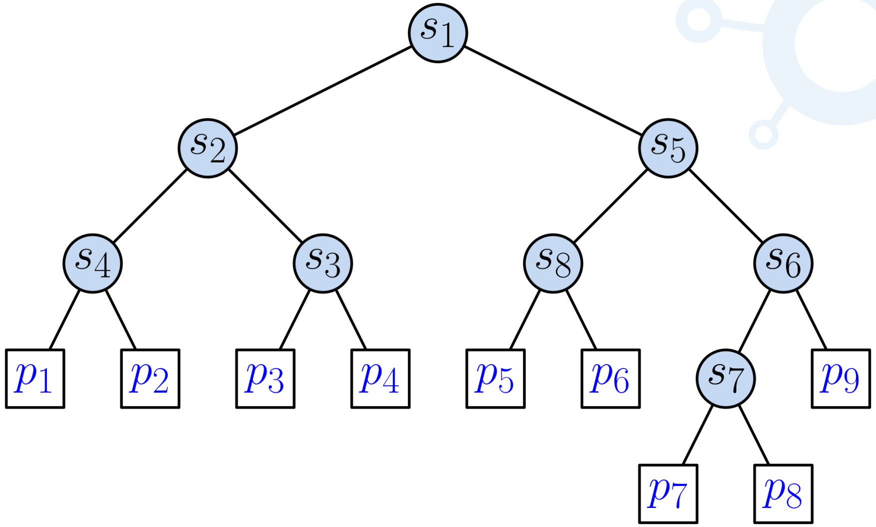
Contents: {2, 6, 7, 9, 11, 14, 17}

Extended Binary Search Trees

Why?

- Memory locality: We saw with B+ trees, we can store many splitters in a single node, increasing fan-out, thus decreasing tree height
- Heterogenous data: In some applications the data and splitters are different
  - Example: Binary-space partition tree
    - Data are points
    - Splitters are lines

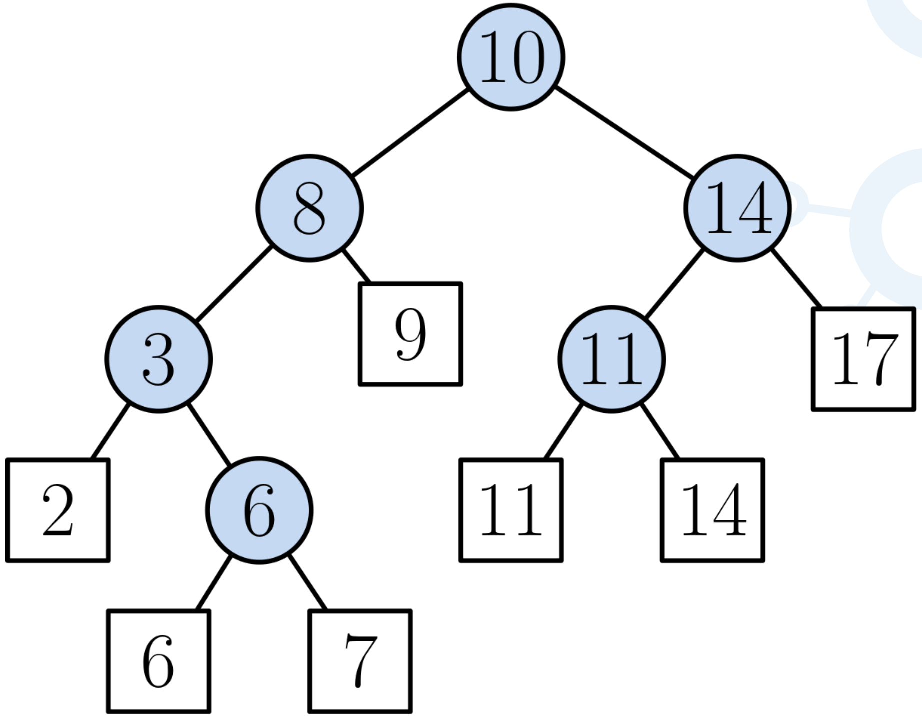




Extended Binary Search Trees

Differences with standard (unbalanced) binary search trees

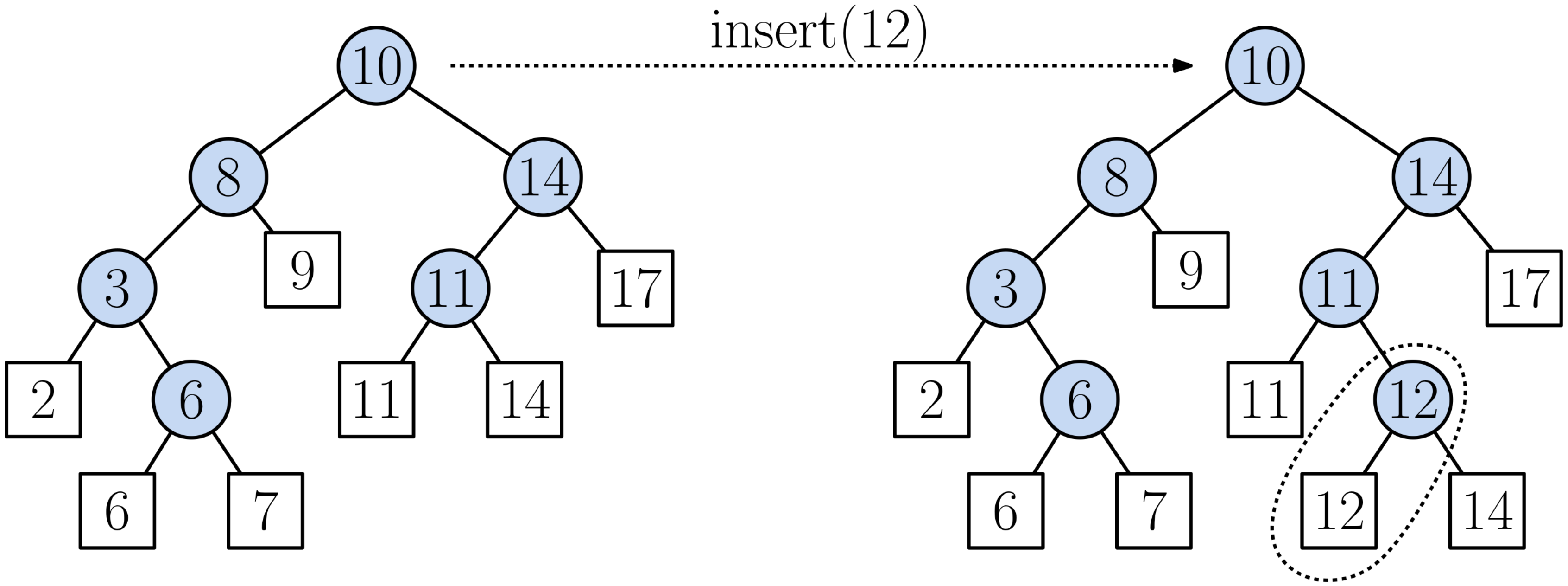
- `find(x)`:
  - Descend to the external node, as directed by internal nodes
  - If key matches - then found, else not
  - Warning: Matching a splitter means nothing!
- Example:
  - `find(7)` - yes
  - `find(15)` - no
  - `find(10)` - no! (even though root matches)



Extended Binary Search Trees

Differences with standard (unbalanced) binary search trees

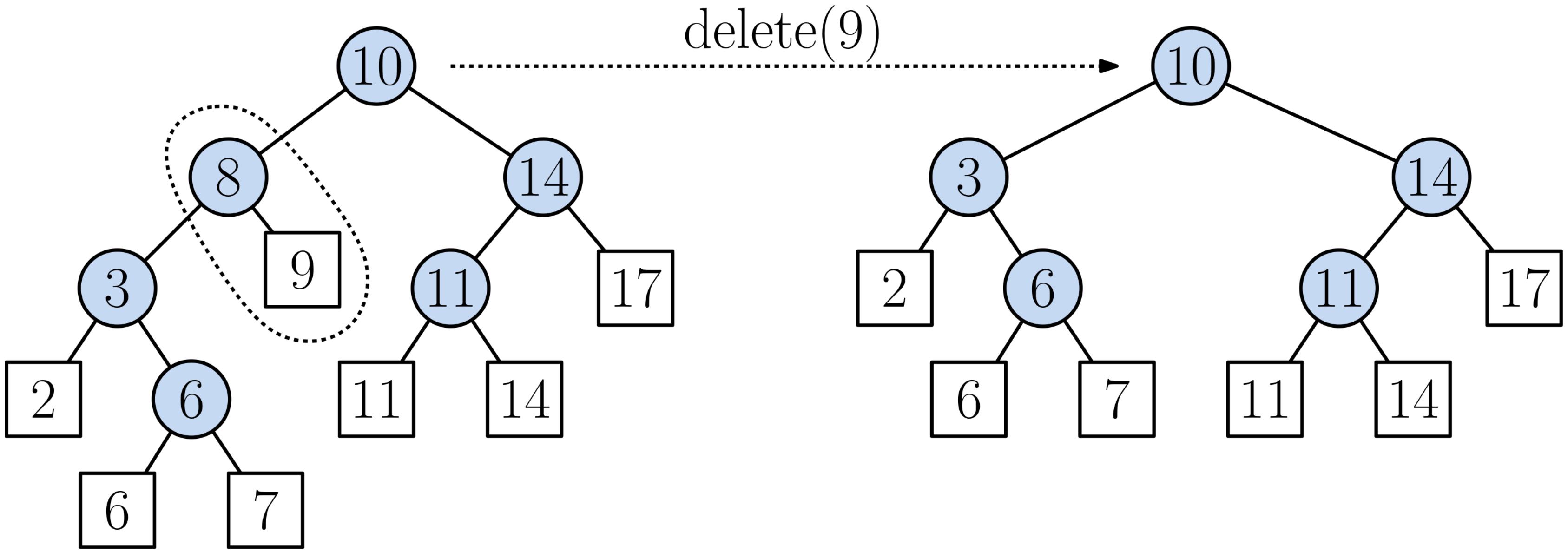
- `insert(x,v)`:
  - Descend to the external node. Let y be its key. If $x = y$ - duplicate-key error
  - Create a new external node for x and internal node to split between x and y
  - Splitter s satisfies: $\min(x, y) \leq s < \max(x, y)$



Extended Binary Search Trees

Differences with standard (unbalanced) binary search trees

- `delete(x)`:
  - Descend to the external node. Let y be its key. If $x \neq y$ - key-not-found error
  - Replace this node and its parent with its sibling



Scapegoat Trees

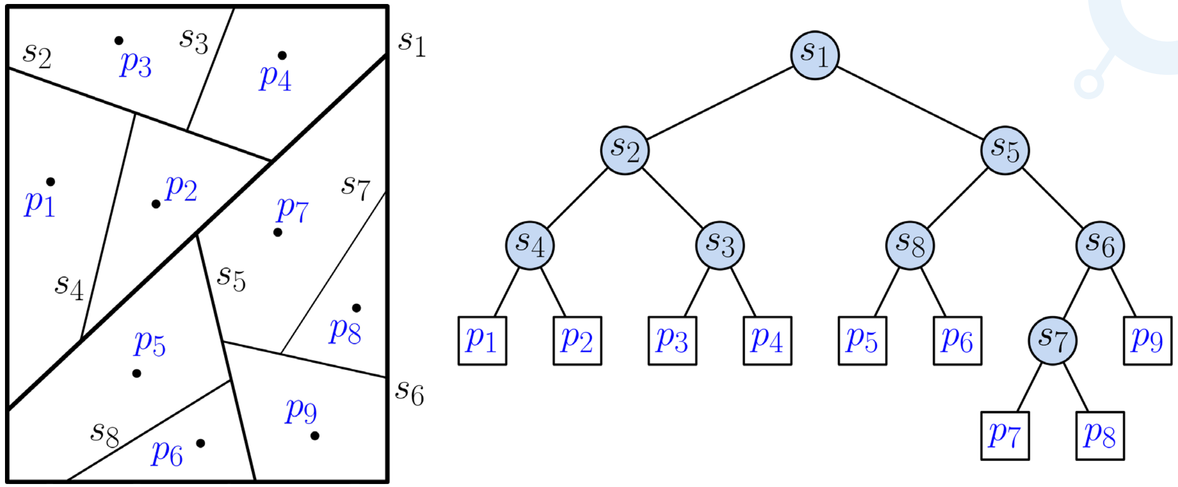
Another Amortized Dictionary Data Structure

- Amortized cost –
  - The total cost divided by the number of operations
  - Splay trees – Amortized cost $O(\log n)$ for dictionary operations, even though any single operation may take $O(n)$ time
- Are there other efficient dictionaries in the amortized sense? Scapegoat trees!
- Origins:
  - Original idea by Arne Andersson (of AA-Tree fame), 1989
  - Rediscovered by Galperin and Rivest, 1993 (gave the name "Scapegoat Tree")
- Resources:
  - http://opendatastructures.org/versions/edition-0.1g/ods-python/8_Scapegoat_Trees.html
  - http://opendatastructures.org/newhtml/ods/latex/scapegoat.html

Scapegoat Trees

Another Amortized Dictionary Data Structure

- Why should we care?
  - Amortized structures are often simpler than worst-case efficient structures
  - The update rules for scapegoat trees can be adapted to many other search trees where rotations cannot be applied (e.g., spatial decomposition trees)
  - The SG Tree in the programming assignment is a variant of the scapegoat tree



Scapegoat Trees

Overview – Balance through Rebuilding

- Insertion:
  - Insert just as in a standard (unbalanced) binary tree
  - Monitor the depth of the inserted node after each insertion
  - If it is too high, then there must be at least one node on the search path that has poor weight balance (left and right children have very different sizes)
  - Find such a node – it's the scapegoat! (It is given the blame for the high depth)
  - Rebuild the subtree rooted at this node so that it is perfectly balanced
- Deletion:
  - Delete as in a standard (unbalanced) binary tree
  - Once the number of deletions is sufficiently large relative to the entire tree size, rebuild the entire tree so it is perfectly balanced

Scapegoat Trees

Overview – Balance through Rebuilding

- How to rebuild a subtree?
  - Perform an inorder traversal of the subtree, and copy the n elements to a (sorted) array $A[0 \ldots n-1]$
  - Take the median of the array as the root, and recursively build left and right subtrees from the two halves of the array
- `buildSubtree(A,i,k)`: Build subtree for k-element subarray $A[i \ldots i+k-1]$
  - If $k = 0$, return null
  - Otherwise, let $m = \left\lfloor \frac{k}{2} \right\rfloor$. Create new node p with median key, $A[i+m]$
  - `p.left = buildSubtree(A,i,m)`
  - `p.right = buildSubtree(A,i+m+1,k-m-1)`
- A subtree with n nodes can be rebuilt in $O(n)$ time

Scapegoat Trees

Overview - Details

- A scapegoat tree stores no balance or height information with the nodes
- In addition to the tree we maintain:
  - n - the current number of nodes in the tree
  - m - an upper bound on the tree size (we maintain: $n \leq m \leq 2n$)
- Height condition: never exceeds $\log_{3/2} m$ ($\Rightarrow$Tree height is $O(\log n)$)
- Size condition:
  - Initially: $n = m = 0$
  - After insertion: $n{+}{+}$, $m{+}{+}$
  - After deletion: $n--$ (but do not change m)
  - If $2n < m$, rebuild the entire tree, and set $m = n$

Scapegoat Trees

Overview – More Details

- `find(x)`:
  - Identical to any binary search time (time: $O(\log n)$)
- delete(x):
  - Identical to delete for an unbalanced binary tree
  - Decrement n (but do not change m)
  - If $2n < m$, rebuild the entire tree, and set $m = n$

Scapegoat Trees

Overview – More Details

- `insert(x)`:
  - Same as standard binary search tree insertion, keep track of inserted node's depth (number of edges from the root)
  - Increment both n and m
  - If inserted depth exceeds $\log_{3/2} m$:
    - Walk back up the search path until we find the first node u such that

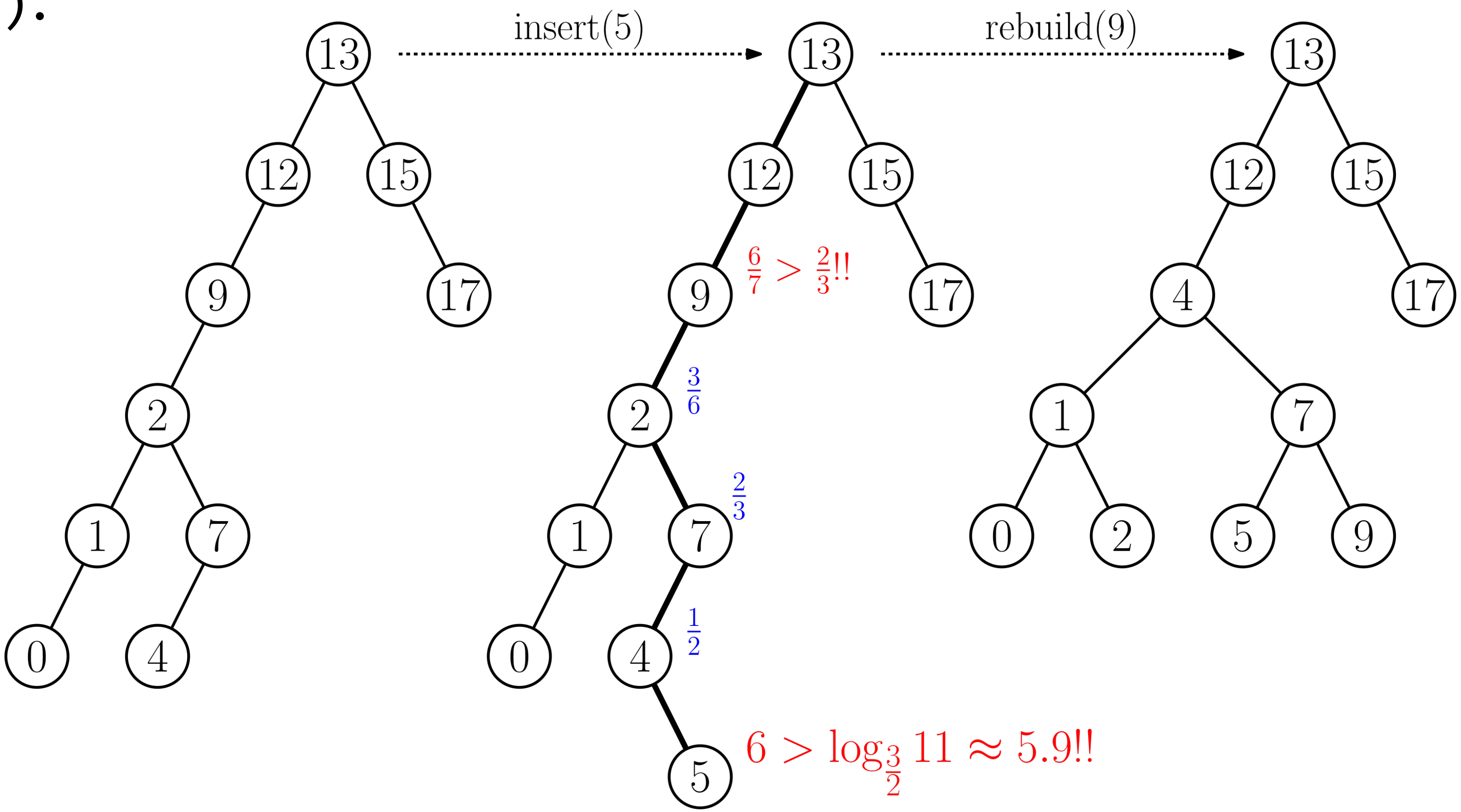
$$\frac{\text{size}(u.\text{child})}{\text{size}(u)} > \frac{2}{3}$$

    - Here $\text{size}(u)$ is the number of nodes in u's subtree and $u.\text{child}$ is u's child on search path
    - A node on the insertion path satisfying this is called a candidate scapegoat
    - Rebuild the subtree rooted at u

Scapegoat Trees

Overview – More Details

- `insert(5):`



Scapegoat Trees

Overview – More Details

- Will we always find a scapegoat node? Yes!
- Is it unique? No! (9, 12, and 13 are all candidate scapegoats)
- Lemma: If there exists a node p such that $\text{depth}(p) > \log_{3/2} m$, then p has an ancestor u that is a candidate scapegoat, that is,

$$\frac{\text{size}(u.\text{child})}{\text{size}(u)} > \frac{2}{3}$$

- Proof: By contradiction.
  - Suppose that for every node u along the path to p, $\text{size}(u.child) \leq (2/3)\text{size}(u)$
  - Letting $k = \text{depth}(p)$, by induction have $\text{size}(p) \leq (2/3)^k\, n$
  - Since $\text{size}(p) \geq 1$, this implies $(3/2)^k \leq n$, implies $k \leq \log_{3/2} n \leq \log_{3/2} m$, contradiction

Scapegoat Trees

Overview – More Details

- How do we compute $\text{size}(u)$ for each node u?
- Two methods:
  1. Maintain a separate field, `u.size`, for each node storing the size of u's subtree (and update as needed)
  2. Compute it on the fly, after each insertion that requires rebalancing:
     - Walk up the search path toward the root
     - Let u be any ancestor of the inserted node. Assume we know $\text{size}(u)$.
     - We want to compute $\text{size}(u.\text{parent})$:
       - Let u' be u's sibling. Traverse the subtree rooted at u' and count the number of nodes.
       - Set $\text{size}(u.\text{parent}) = 1 + \text{size}(u) + \text{size}(u')$
       - This may seem costly, but it can all be done within the amortized time bound!

Scapegoat Trees

Amortized Complexity

- Theorem: Starting with an empty tree, any sequence of k dictionary operations costs a total of $O(k \log k)$
- Proof: (Sketch)
  - Find: Cost is $O(\log n)$ always (by height bound)
  - Delete: In order to rebuild a tree due to deletions, at least half the entries have been deleted. A token-based analyses (recall stacks and rehashing from earlier lectures) can be applied here.
  - Insert: This is analyzed by a potential argument. Intuitively, after any subtree of size k is rebuilt it takes $O(k)$ insertions to force this subtree to be rebuilt again. Charge the rebuilding time against these "cheap" insertions.
- Corollary: The amortized complexity of the scapegoat tree with at most n nodes is $O(\log n)$

SG Tree

A data structure invented just for the programming assignment

- Overview - An SG Tree is:
  - An extended binary search tree that is rebalanced like a scapegoat tree
  - Updated concepts:
    - The size of an internal node is the number of external nodes in its subtree
    - The height of a node is the maximum number of edges to any external node
  - Similarities with the scapegoat tree:
    - Maintain total size n and upper bound m, where $n \leq m \leq 2n$
    - Height condition: Rebuild if tree height exceeds $\log_{3/2} m$ (⇒Tree height is $O(\log n)$)
    - Candidate scapegoat: Any node on search path such that $\text{size}(u.\text{child})/\text{size}(u) > \frac{2}{3}$
    - Deletion condition: If $2n<m$, rebuild the entire tree, and set $m=n$

SG Tree

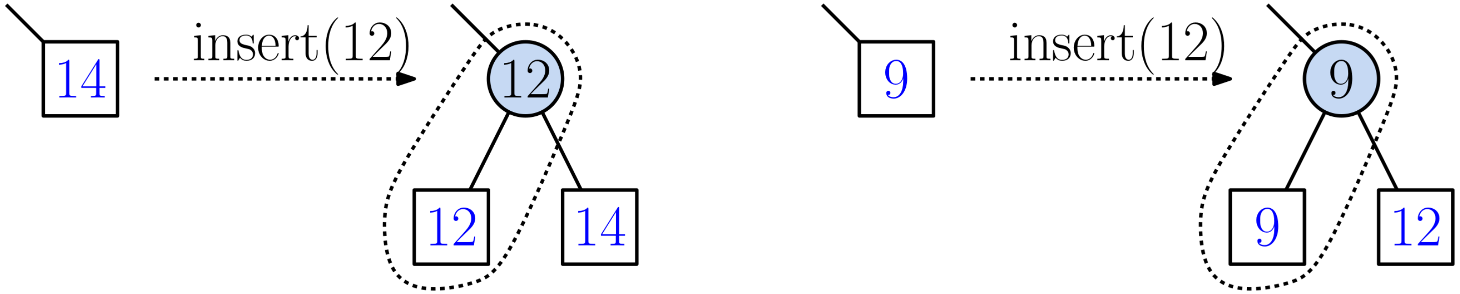
- Differences from the scapegoat tree:
  - Nodes: Two types of nodes:
    - External - store data only (a city for the programming assignment)
    - Internal - store splitter, left, right, subtree height, and subtree size
  - Scapegoat Node:
    - When insertion causes the tree's height to exceed $\log_{3/2} m$, if multiple nodes satisfy the scapegoat condition, we chose the one closest to the root
    - Why? By rebuilding the largest subtree, we make the overall tree more balanced

SG Tree

- Conventions:
  - To avoid floating-point round-off errors, use integer arithmetic to test the scapegoat condition:

$$2 \cdot \text{size}(u) < 3 \cdot \text{size}(u.\text{child}) \implies u \text{ is candidate scapegoat}$$

  - When inserting a new external node, the parent's splitter is taken from its left child



SG Tree

- More conventions:
  - When rebuilding a subtree with k external nodes:
    - If k is even, split the internal nodes evenly among the left and right subtrees
    - If k is odd, the left subtree gets $\lceil k/2 \rceil$ external nodes and the right subtree gets $\lfloor k/2 \rfloor$
    - More formally: When splitting the k-element subarray $A[i \ldots i+k-1]$:
      - Set $m = \lceil k/2 \rceil$
      - Build left subtree with m keys: $A[i \ldots i+m-1]$
      - The splitter is $A[i+m-1]$
      - Build right subtree with $k-m$ keys: $A[i+m \ldots i+k-1]$
    - This convention results in the most even split and most balanced splitter value

SG Tree

Implementation hints

- Abstract class Node and two derived classes
  - ExternalNode - stores just a city object
  - InternalNode - stores splitter (a city), left, right, size, and height
- Take advantage of virtual functions when defining node operations
  - Don't do this:

```
Node insert(Node p) {
        if (p.isExternal) {
                ExternalNode pe = (ExternalNode) p;
                /* external node processing */
        }
        else {
                InternalNode pi = (InternalNode) p;
                /* internal node processing */
        }
}
```

SG Tree

Implementation hints

- Instead, do this:

```
abstract class Node {
   // …
   abstract Node insert(Key x);
}
class InternalNode extends Node {
   // …
   Node insert(Key x) { … } // insertion at internal node
}
class ExternalNode extends Node {
   // …
   Node insert(Key x) { … } // insertion at external node
}
```

SG Tree

Implementation hints

- Your SGTree class:
  - Generic? It’s up to you.
    - We don’t maintain key-value pairs. We store city objects.
    - The print command assumes that the data object has a name and (x,y) coordinates
    - We made ours generic, but the data type must support `getName()`, `getX()`, and `getY()`
  - Use inner classes for nodes:
    - `Node, InternalNode, ExternalNode`
  - Private data:

```
Node root;
Comparator comparator; (Optional. Given with the constructor)
Document resultsDoc; (Needed by print command)
int n, m; (Used by the scapegoat functions)
```

SG Tree

Implementation hints

- `insert(x)`:
  - Insert the key using the standard recursive insertion algorithm
    - Some modifications needed because we have an extended tree
  - While backing out from recursion, update the size and height values for each node
  - Increment both n and m
  - If the new tree height exceeds $\log_{3/2} m$:
    - Traverse the search path from root down until finding the first candidate scapegoat

$$2 \cdot \text{size}(u) < 3 \cdot \text{size}(u.\text{child})$$

    - Rebuild this subtree (Note: u must be an internal node)
    - (Be sure that your rebuilding function updates heights and sizes for all nodes)

SG Tree

Implementation hints

- `delete(x)`:
  - Delete the key using the standard recursive deletion algorithm
    - Some modifications needed because we have an extended tree
  - While backing out from recursion, update the size and height values for each node
  - Decrement n but not m
  - If $2n < m$:
    - Rebuild the entire tree
    - Set $m = n$

SG Tree

Implementation hints

- Write utilities for handling size and height:
  - `getSize(Node p):     return (p.isExternal ? 1 : p.size)`
  - `getHeight(Node p):   return (p.isExternal ? 0 : p.height)`
  - `InternalNode.update():`

```
size = getSize(left) + getSize(right);
height = 1 + max(getHeight(left), getHeight(right));
```

- Write a debugging utility for "pretty printing" your tree
  - Call this function after each major operation (insert, delete, subtree rebuilding)
- Insert a boolean flag (e.g., DEBUG), which you can turn on and off for debugging

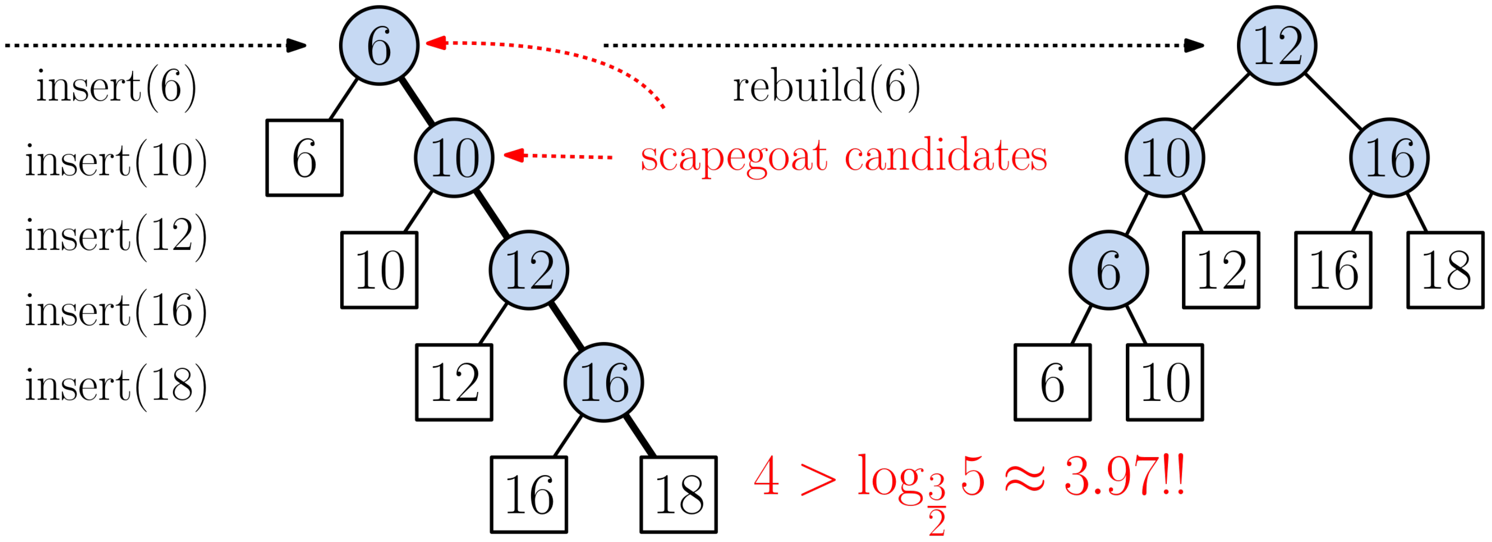
SG Tree

Implementation hints

- Problem:
  - My SG Tree is ordered by (x, y)-coordinates. How do I delete a city given just its name?
- Answer:
  - This is why we have the binary-search tree (which is ordered by name)
  - Create a "bogus city" with just a name (no coordinates)
  - Find this city in your binary-search tree and save this "complete city"
  - Delete this complete city from both data structures

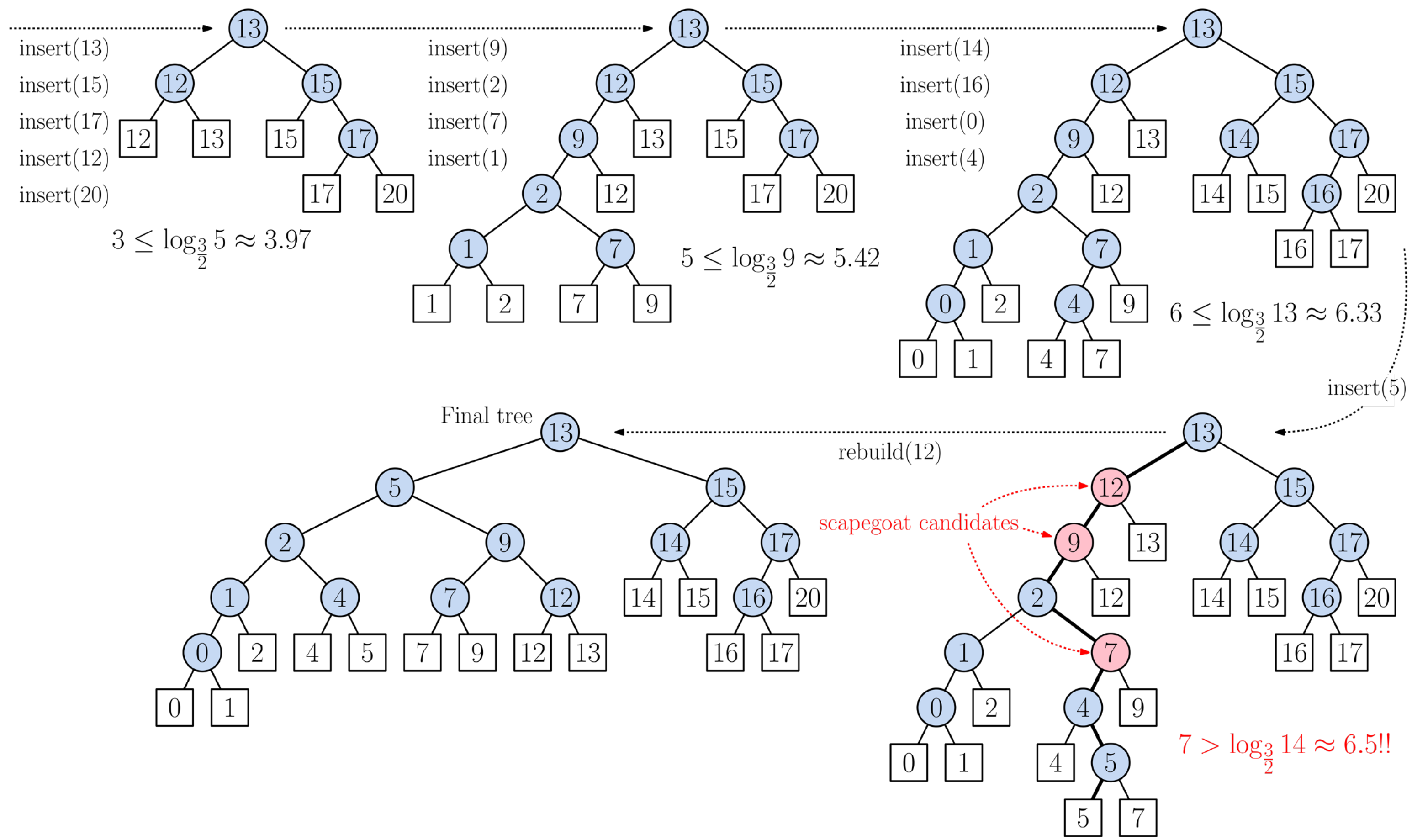
Supplemental

Example of a rebuild operation



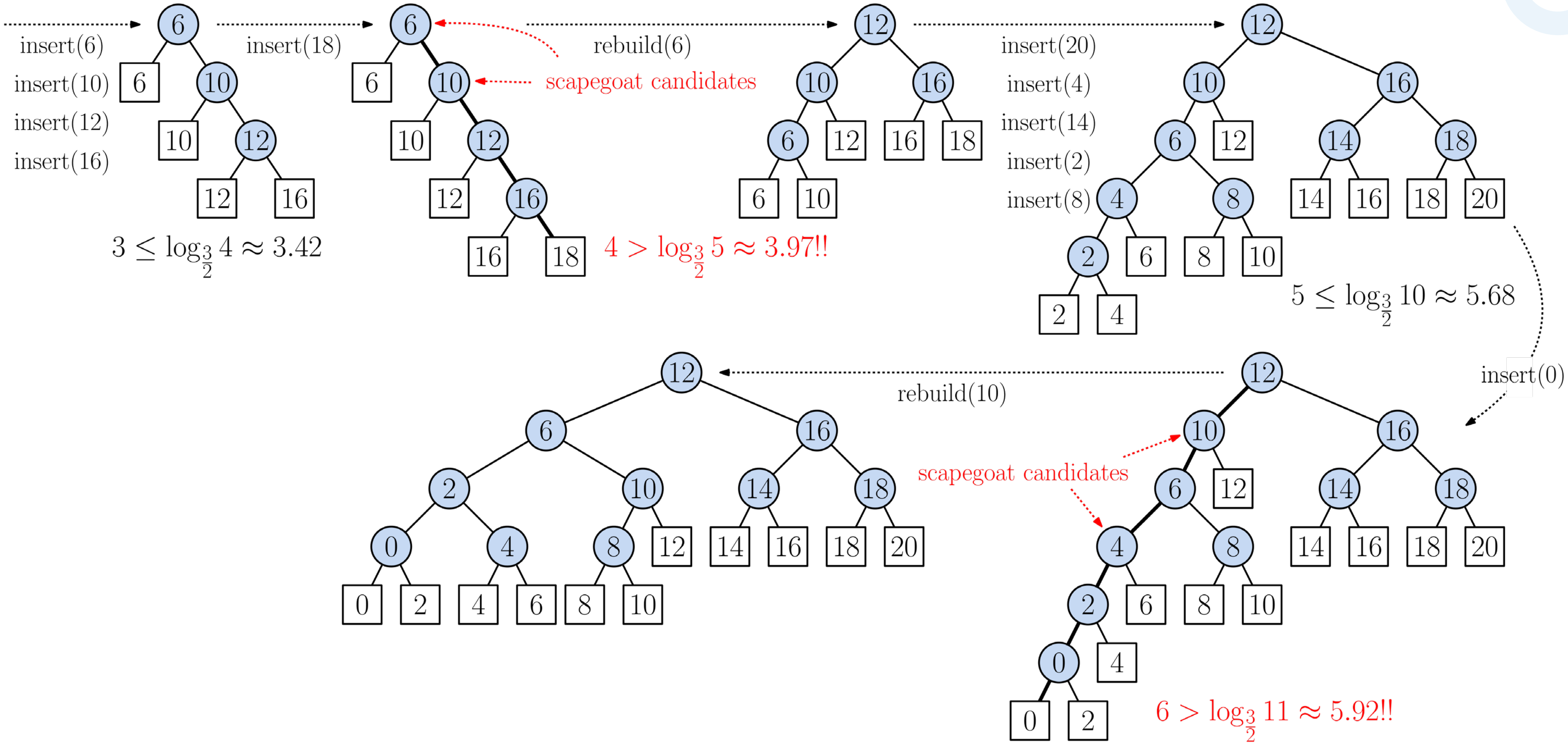
Supplemental

Example of SG-Tree operations



Supplemental

Another example of SG-Tree operations



Summary

- Extended Binary Search Trees
  - Data stored only at the leaves (external nodes)
  - Internal nodes are used only for locating the data
- Scapegoat Trees
  - Another amortized binary search tree data structure
  - Rebalancing through rebuilding subtrees
  - Unlike splay trees, height is guaranteed to be $O(\log n)$
- SG Tree
  - An extended-tree variant of the scapegoat tree