CMSC 420 – 0201 – Fall 2019
Lecture 13

Point Quadtree and kd-Trees

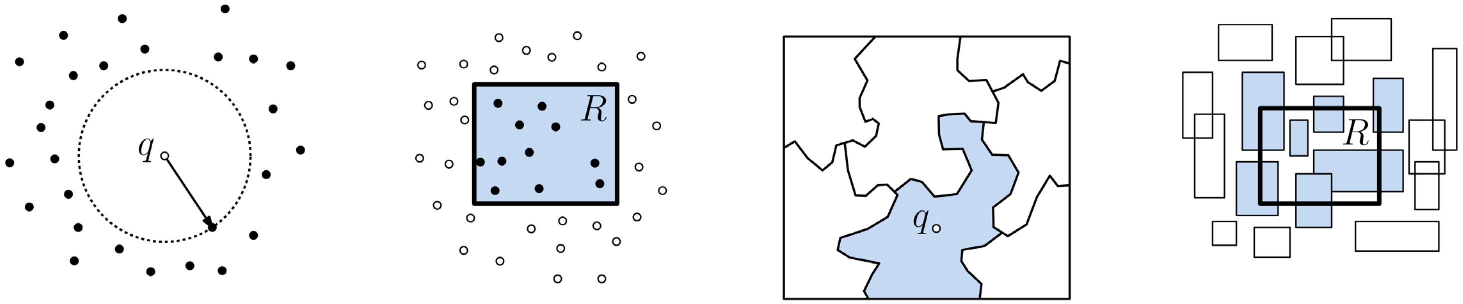
Overview

- So far, we have studying data structure for 1-dimensional search problems
- Many data structure problems involve data in multi-dimensional spaces:
  - Spatial databases, automated cartography (maps), and navigation
  - Computer graphics
  - Robotics and motion planning
  - Solid modeling and industrial engineering
  - Particle and fluid dynamics
  - Molecular dynamics in computational biology
  - Machine learning
  - Image processing, pattern recognition, computer vision

Geometric Queries

Examples

- Nearest-neighbor searching – Find the closest point to a given query point q
- Range searching – Report/Count the points lying within a query region R
- Point location – Find the region of a subdivision (map) containing a query point q
- Intersection searching – Find all the objects that overlap a given query object R
- Ray shooting – Find the first (if any) object hit by shooting a ray from a point p



(a) (b) (c) (d)

Geometric Queries

Similarities and differences

- Multi-dimensional data structures borrow many concepts from 1-dimensional search structure
  - Tree-based structures based on hierarchical partitions
  - Maintaining balance O(log n) height
  - Use key/splitters to navigate the search space
- Many differences as well
  - There is no natural total order in geometric space.
  - What does it mean to say one point is smaller than another?

Geometric Data

Point Representation

- A point in a d-dimensional space is represented by a d-vector of reals:

$$p = (p_1, p_2, \dots, p_d)$$

- In Java, this could be represented by a d-element array

```
float[] p = new float[d];
```

- While in linear algebra, indexing is from $1 \dots d$, in Java indexing is from $0 \dots d-1$
- A set of n points can be represented as a 2-dimensional array:

```
float[][] P = new float[n][d];
```

Geometric Data
Point Representation

- A better approach is to encapsulate points in a class structure

```
public class Point {
    private float[] coord;                  // coordinate storage

    public Point(int dim) {  /* construct a zero point */ }
    public int getDim() { return coord.length; }

    public float get(int i) { return coord[i]; }
    public void set(int i, float x) { coord[i] = x; }

    public boolean equals(Point other) { /* compare with another point */ }
    public String toString() { /* convert to string */   }
}
```

Point Quadtree

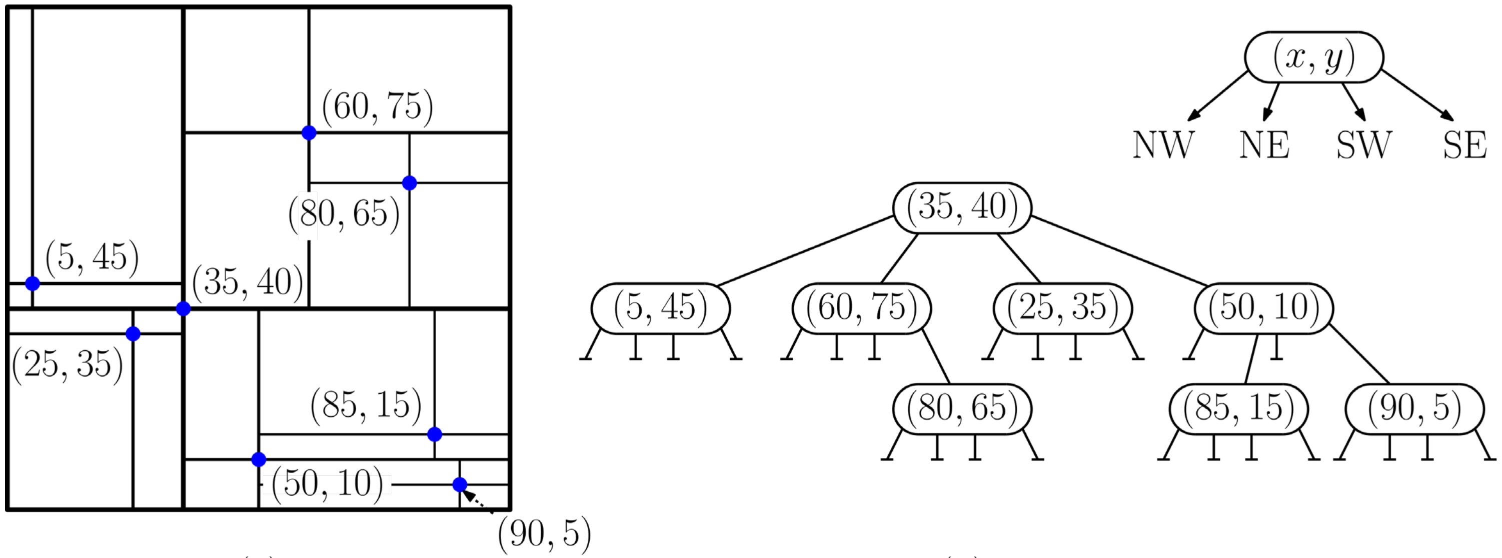
A Natural Generalization of Binary Search Trees

- How do we generalize a 1-dimensional tree to d-dimensional space?
- Partition tree:
  - Each node is associated with a region of space (e.g., a rectangle), its cell
  - Each internal node contains a splitter, which subdivides space into smaller regions
  - Data may be stored in the nodes (as the splitters) or in external nodes (as in extended binary search trees)
- Point Quadtree:
  - Each node stores a point (both data and splitter)
  - 2-dimensions: Horizontal and vertical lines through point subdivide cell into 4 quadrants
  - d-dimensions: d axis-parallel hyperplanes passing through the point subdivide space into 2^d (generalized) orthants
  - Each node has 2^d (possibly null) children

Point Quadtree

A Natural Generalization of Binary Search Trees

- In 2D, quadrants are labeled NW, NE, SW, and SE
  - Example: (35,40), (50,10), (60,75), (80,65), (85,15), (5,45), (25,35), (90,5)
- To locate a point, we descend from the root, visiting the appropriate child



(a) (b)

Point kd-Tree

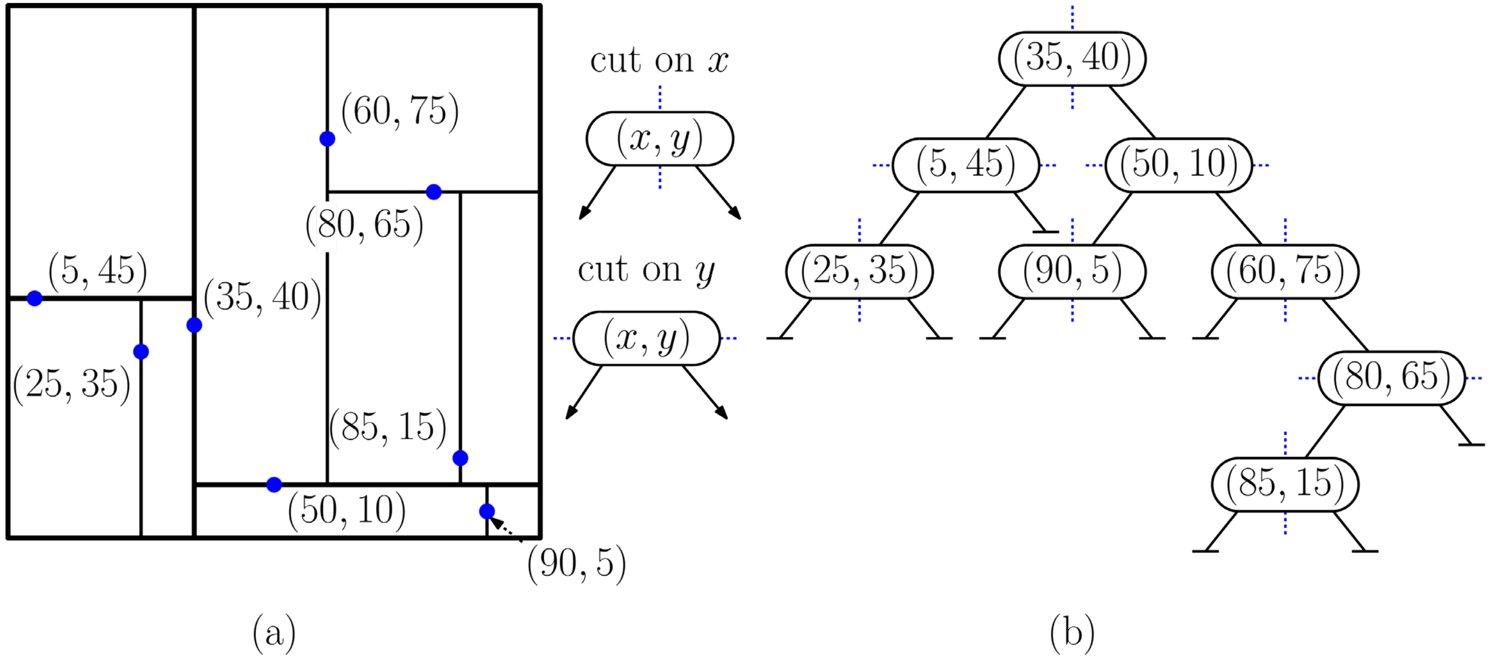
A Binary Partition Tree

- The point quadtree works fine in low-dimensional space, but does not scale well to high dimensional space. For example, in $d = 20$ space, each node has a fanout of $2^d \approx 1{,}048{,}576$
- Idea: Let's just split one dimension at a time
- Point kd-tree:
  - Each node stores a point (both data and splitter)
  - And an index i, $0 \leq i \leq d-1$, the cutting dimension
  - For any point $x = (x_0, \ldots, x_{d-1})$:
    - If $x_i < p_i$, x goes in the left subtree
    - If $x_i \geq p_i$, x goes in the right subtree
  - Cutting dimension varies by level (e.g., `p.child.cutDim = (p.cutDim+1)%dim`)

Point kd-Tree

A Binary Partition Tree

- Example: (35,40), (50,10), (60,75), (80,65), (85,15), (5,45), (25,35), (90,5)
- Cutting dimension alternates between x and y



Point kd-Tree

Node structure

```
class KDNode {                    // node in a kd-tree
    Point point;                  // splitting point
    int cutDim;                   // cutting dimension
    KDNode left;                  // children
    KDNode right;

    KDNode(Point point, int cutDim) {    // constructor
        this.point = point;
        this.cutDim = cutDim;
        left = right = null;
    }

    boolean inLeftSubtree(Point x) {     // is x in left subtree?
        return x.get(cutDim) < point.get(cutDim);
    }
}
```

Point kd-tree

Point insertion

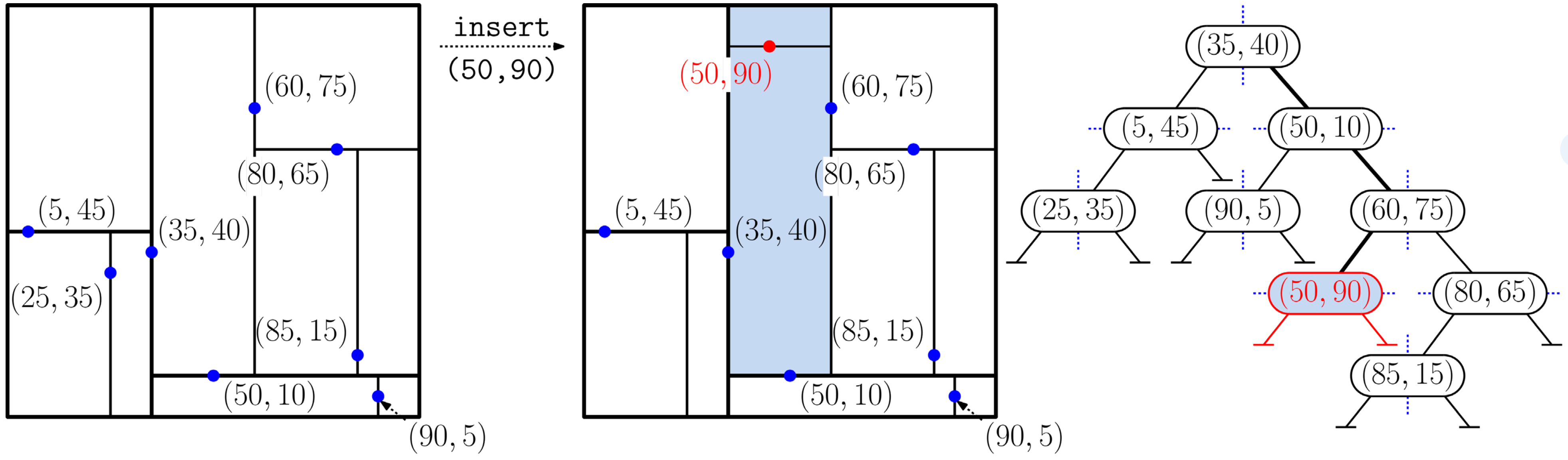
- To insert a point, descend the tree to find the leaf cell containing the point
- Create a new cell and assign its cutting dimension

```
KDNode insert(Point x, KDNode p, int cutDim) {
    if (p == null) {                              // fell out of tree
        p = new KDNode(x, cutDim);                // create new leaf
    } else if (p.point.equals(x)) {
        throw Exception("duplicate");             // duplicate data point!
    } else if (p.inLeftSubtree(x)) {              // insert into left
        p.left = insert(x, p.left, (p.cutDim + 1) % x.getDim());
    } else {                                      // insert into right
        p.right = insert(x, p.right, (p.cutDim + 1) % x.getDim());
    }
    return p;
}
```

Point kd-Tree

Point insertion

- `Insert(50,90):`



Point kd-tree

Point deletion

- Deletion is more complicated – Need a s node
- How to choose the replacement?
  - Can't just take the inorder successor (inorder doesn't make geometric sense)
  - Depends on the current cutting dimension i
  - Want the point of the right subtree with the minimum i coordinate $p[i]$
- Utility: Select the point with the smaller ith coordinate

```
Point minAlongDim(Point p1, Point p2, int i) { // return smaller point on dim i
    if (p2 == null || p1[i] <= p2[i])
        return p1;
    else
        return p2;
}
```

Point kd-tree

Utility for finding replacement nodes

- Utility: Find the point that minimizes ith coordinate in subtree p
  - `if (p.cutDim == i):`
    - The subtrees are ordered by the ith coordinate
    - Look recursively in p's left subtree, if it exists
    - If not, take `p.point`
  - `if (p.cutDim != i):`
    - The subtrees are ordered arbitrarily with respect to I
    - Compute the minima from p's left and right subtrees recursively
    - Use findMin to select the overall minimum from left-min, right-min, and `p.point`

Point kd-tree

Utility for finding replacement nodes

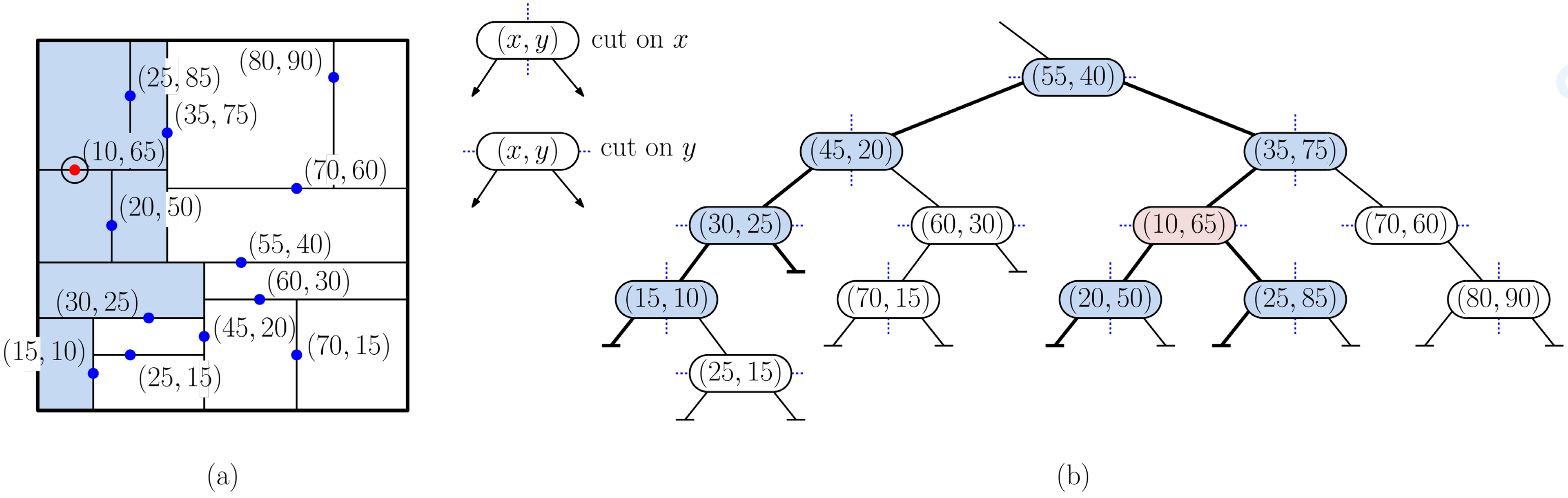
- Utility: Find the point that minimizes ith coordinate in subtree p

```
Point findMin(KDNode p, int i) {                          // get min point along dim i
    if (p == null) {                                      // fell out of tree?
        return null;
    }
    if (p.cutDim == i) {                                  // cutting dimension matches i?
        if (p.left == null)                               // no left child?
            return p.point;                               // use this point
        else
            return findMin(p.left, i);                    // get min from left subtree
    } else {                                              // it may be in either side
        Point q = minAlongDim(p.point, findMin(p.left, i), i);
        return minAlongDim(q, findMin(p.right, i), i);
    }
}
```

Point kd-tree

Utility for finding replacement nodes

- Example: Find minimum along x
  - If cut dim = x: Try left child (or p itself)
  - If cut dim = y: Try both children



Point kd-tree

Point deletion

- Overview: Delete x from subtree p
  - `if (p == null):`
    - Fell out of the tree - Error: attempt to delete nonexistent point!
  - `else:`
    - If both of p's children are null - Simply unlink p (return null)
    - If p's right child exists:
      - Invoke `findMin(p.right, p.cutDim)` to compute replacement node
      - Copy its contents to p
      - Recursively delete the replacement node from `p.right`
    - Else:
      - Tricky!

Point kd-tree

Point deletion

- Overview: Delete x from subtree p, where p has a left child but no right child:
  - In the 1D case, we just unlinked p
    - But this has the effect of promoting p’s child up a level
    - The cutting dimensions no longer cycle from parent to child. (Do we care? Suppose we do)
  - How about picking the maximum point in p’s left subtree?
    - Our tie-breaking rule assumed that points in the left subtree have coordinates strictly smaller than the splitter
    - This will cause problems if there are duplicate coordinates in p’s left subtree
  - Final answer (very sneaky!)
    - Compute the minimum from p’s left subtree as replacement (But it’s on the wrong side!)
    - Make the left subtree the new right subtree. (Amazingly, this works!)

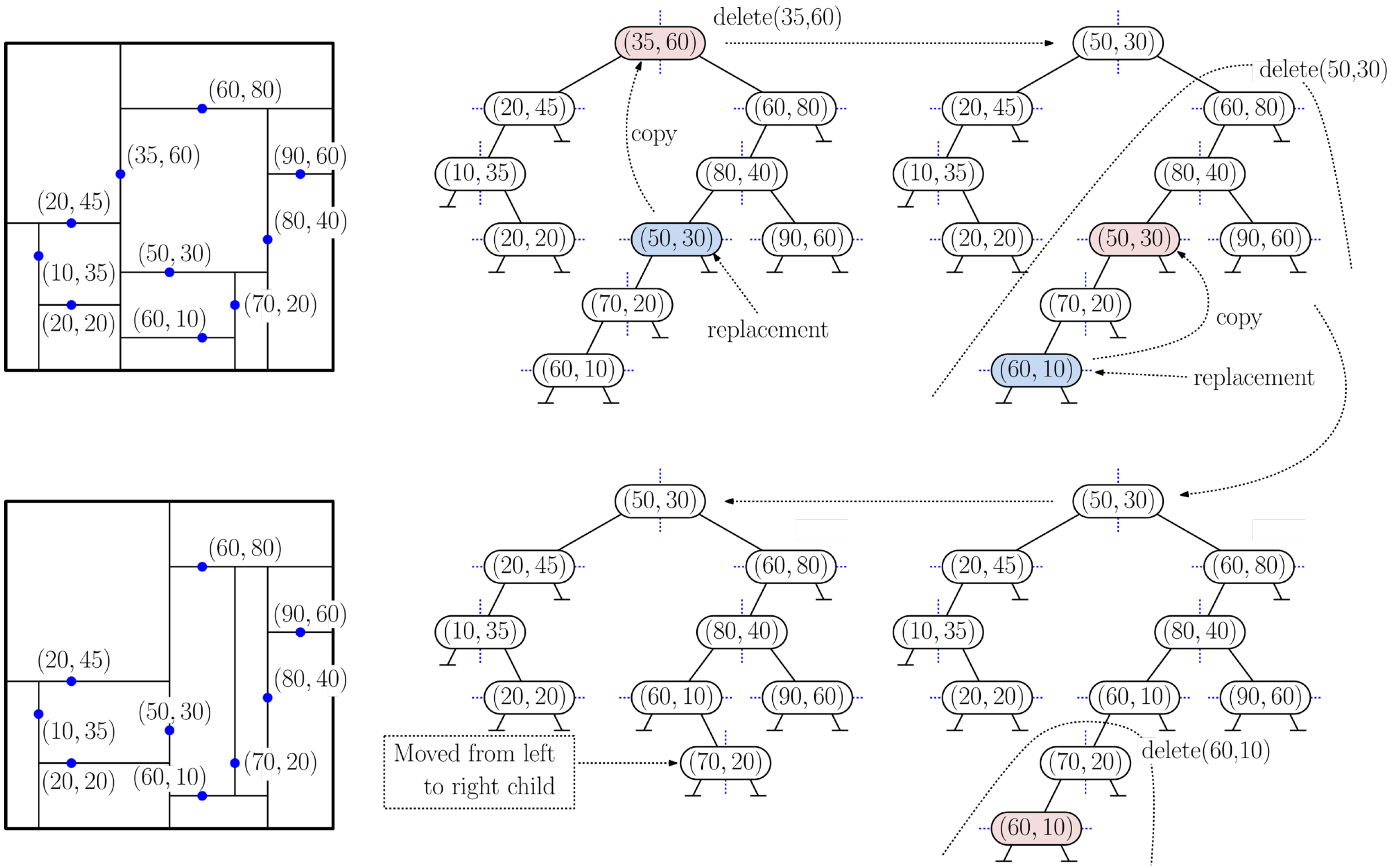
Point kd-tree

Point deletion

```
KDNode delete(Point x, KDNode p) {
    if (p == null) {                                      // fell out of tree?
        throw Exception("point does not exist");
    } else if (p.point.equals(x)) {                       // found it
        if (p.right != null) {                            // take replacement from right
            p.point = findMin(p.right, p.cutDim);
            p.right = delete(p.point, p.right);
        } else if (p.left != null) {                      // take replacement from left
            p.point = findMin(p.left, p.cutDim);
            p.right = delete(p.point, p.left);            // move left subtree to right!
            p.left = null;                                // left subtree is now empty
        } else {                                          // deleted point in leaf
            p = null;                                     // remove this leaf
        }
    } else if (p.inLeftSubtree(x)) {
        p.left = delete(x, p.left);                       // delete from left subtree
    } else {                                              // delete from right subtree
        p.right = delete(x, p.right);
    }
    return p;
}
```

Point kd-tree

Point deletion - Example



Point kd-tree

Analysis

- Analogous to unbalanced binary search trees
  - Storage space linear in n, the number of points
  - All dictionary operations (insert, delete, find) take time proportional to tree's height
  - Theorem: If n points are inserted in random order, the expected height of the kd-tree is $O(\log n)$
  - I'd conjecture that deletion suffers from the same systematic bias, which would lead to heights of $\sqrt{n}$ after long sequences of random insertions and deletions, but I know of no results from the literature

Summary

- Geometric Search
  - Point representation
- Point Quadtree
- Point kd-Trees
  - Node representation (point and cutting dimension)
  - Insertion
  - Deletion
    - FindMin utility
    - Sneaky trick to compute replacement nodes
  - Analysis: $O(\log n)$ time assuming random insertions