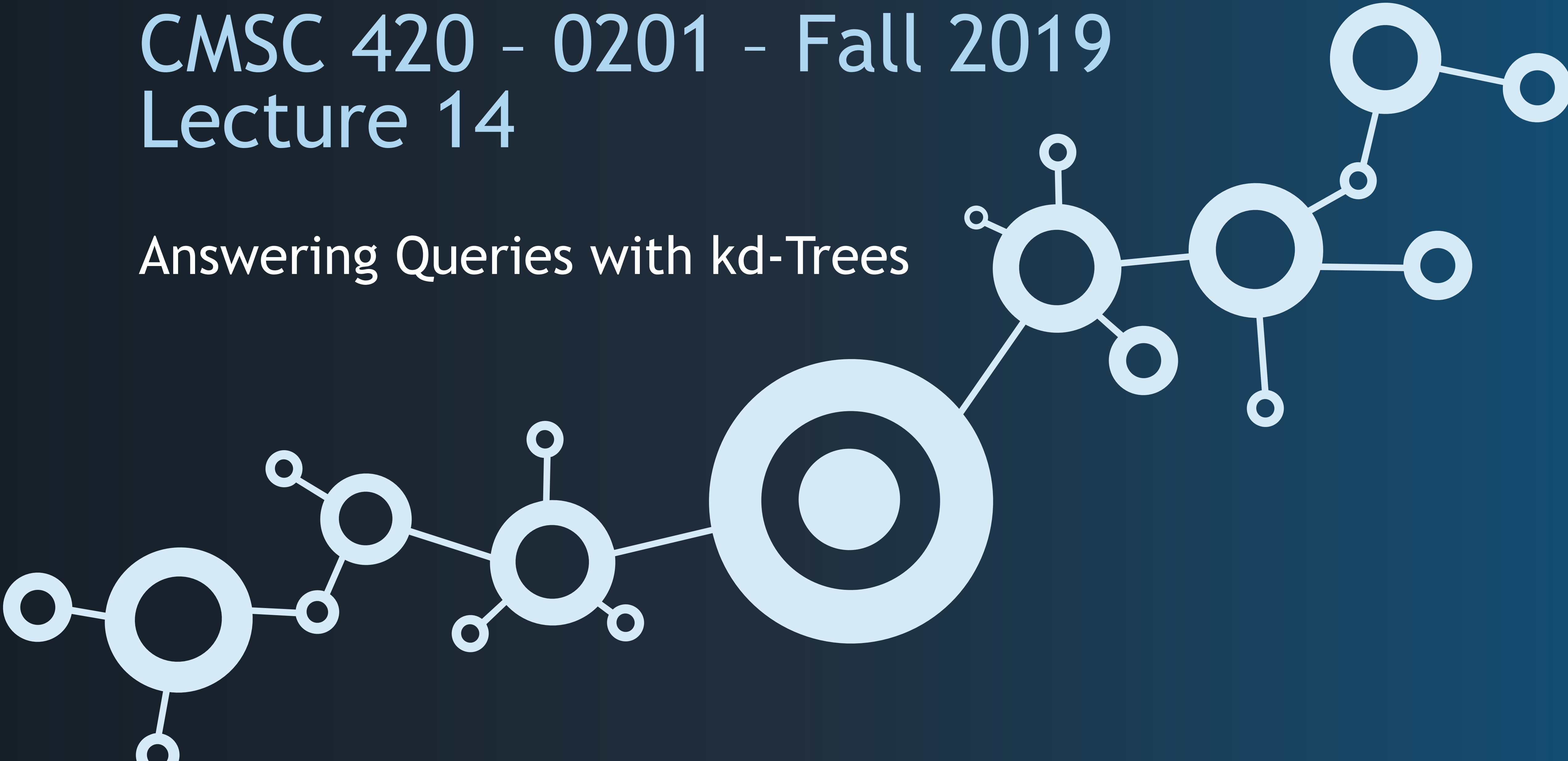
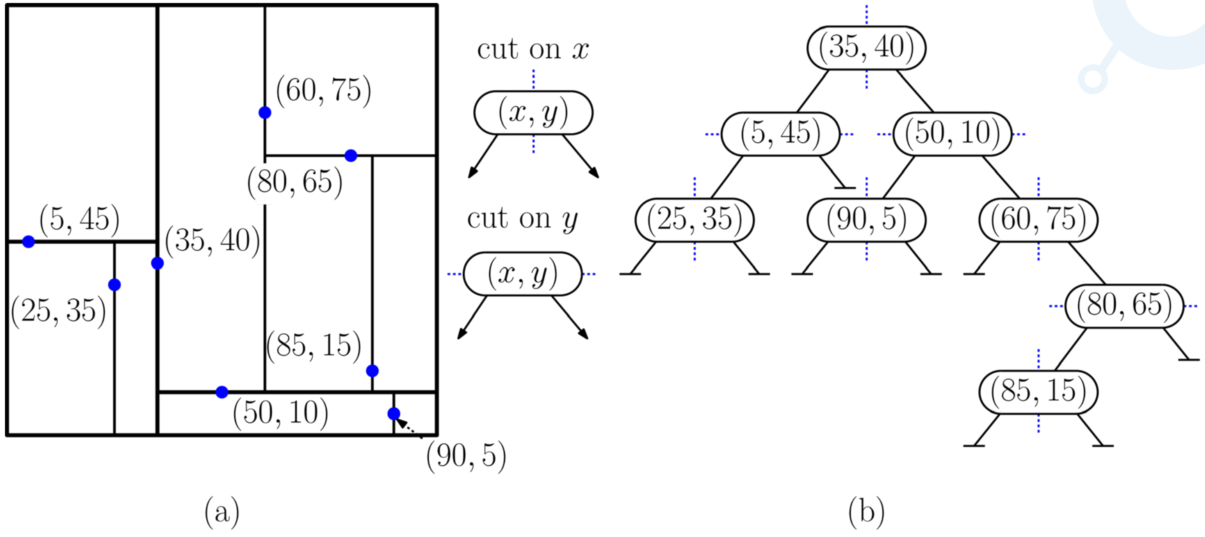
CMSC 420 – 0201 – Fall 2019
Lecture 14

Answering Queries with kd-Trees



Overview

- Previously, we introduced the kd-tree, a spatial binary partition tree:
  - Stores a set of points in d-dimensional real space, where each point p is represented as a d-element Java vector `p[0,…,d-1]`
  - Each node stores a point p and a cutting dimension i, where $0 \leq i \leq d-1$
  - The left subtree contains points x such that $x[i] < p[i]$ and the right contains points such that $x[i] \geq p[i]$
  - Cutting dimension varies from node to node (e.g., cycles from 0 through $d-1$, but other strategies are possible)
  - What other queries can we answer?



Overview

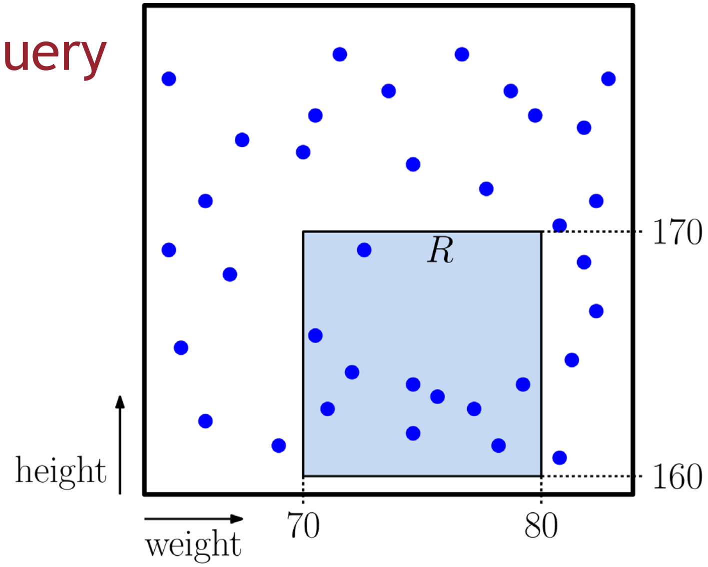
Queries

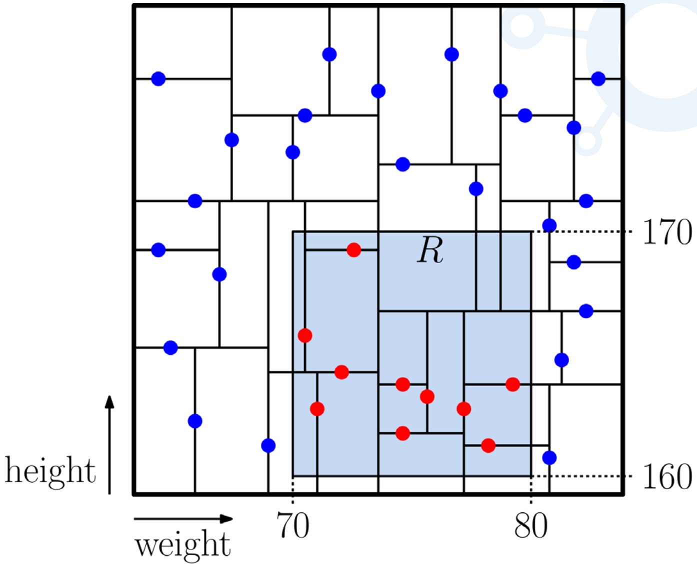
- Orthogonal range query:
  - Given a point set P stored in a kd-tree, a query consists of a d-dimensional axis parallel rectangle R
  - Range counting query: How many points of P lie within R?
  - Range reporting query: Report all the points of P that lie within R. (Java: Return an iterator for the set $P \cap R$)
- Nearest-neighbor query:
  - Given a point set P stored in a kd-tree, a query consists of a point q
  - Nearest-distance query: What is the distance to q's closest point in P
  - Nearest-neighbor query: Report the point that is closest to q
  - k-th Nearest-neighbor query: Report the k closest points of P to q

Overview

Queries

- Orthogonal range queries
  - Given a medical database. Each patient associated with a vector of biomedical statistics (weight, height, blood pressure,…)
  - Want to count the number of patients whose weight, height, BP, etc. are within a given range of values
  - This is range counting query

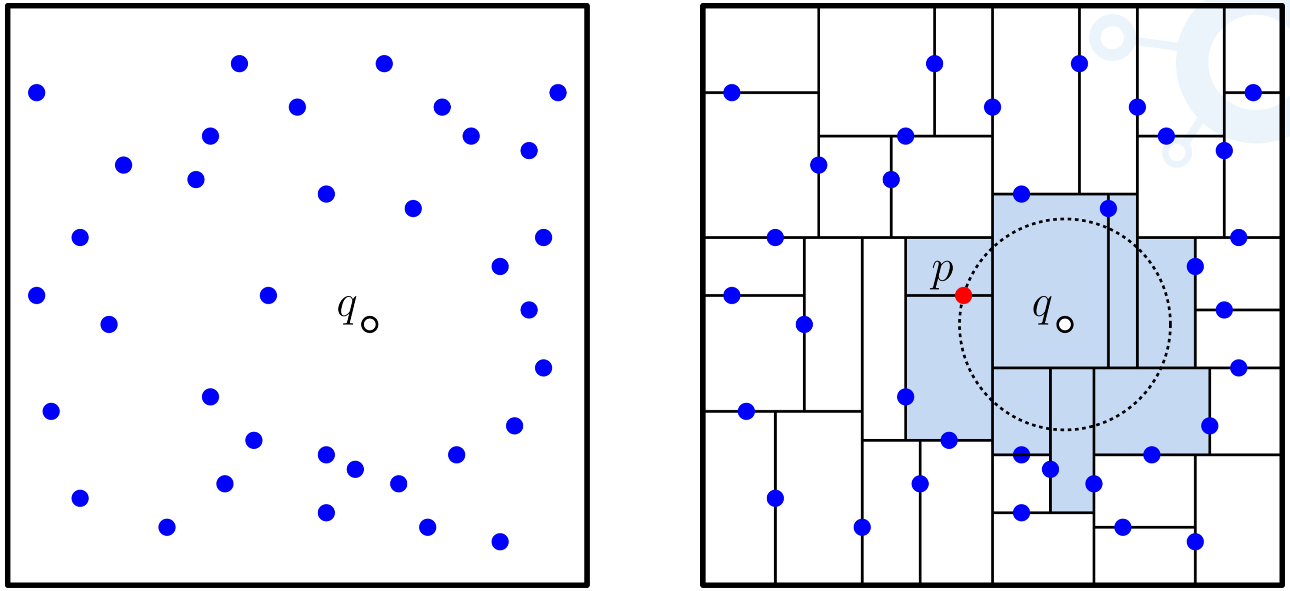




Overview

Queries

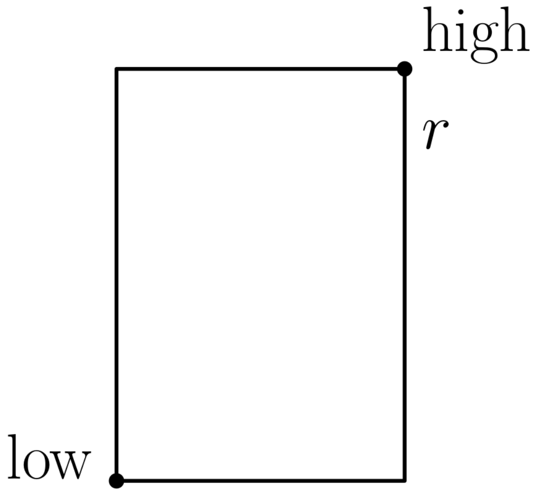
- Nearest-neighbor queries
    - In a large database of documents, each document is encoded as a vector describing document properties (e.g., trigrams: number of occurrences of triples of characters)
    - Given a sample document q, we want to find similar documents in the database
    - This is a nearest-neighbor query



Orthogonal Range Queries

A Rectangle Class

- d-dimensional axis-aligned (hyper-)rectangles are useful geometric objects
- Rectangle class:
  - Defined by two d-dimensional points, low and high
  - The rectangle consists of the points q, such that $\mathrm{low}[i] \leq q[i] \leq \mathrm{high}[i]$, for $0 \leq i \leq d-1$

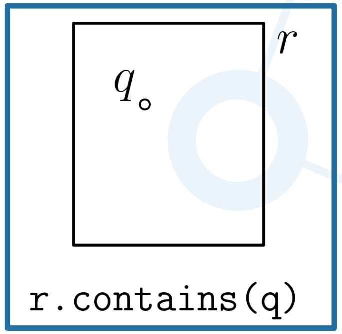


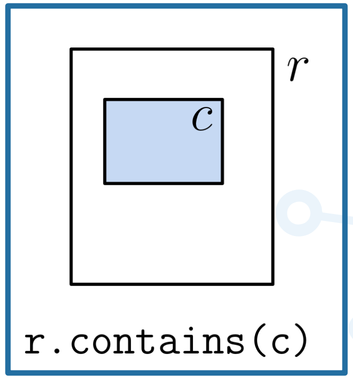
`new Rectangle(low, high)`

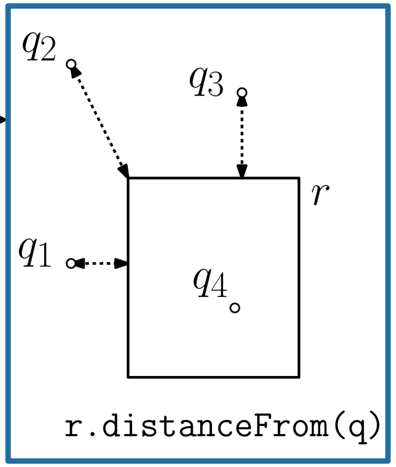
Orthogonal Range Queries

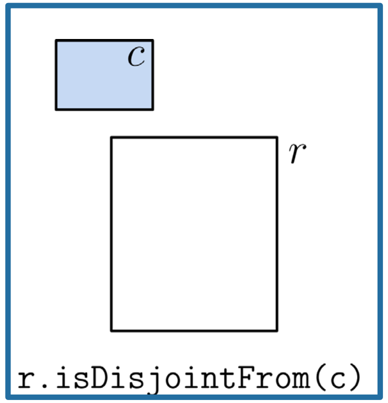
A Rectangle Class

- Some useful functions:
  - `r.contains(Point q)`: true if r contains point q
  - `r.contains(Rectangle c)`: true if r contains rectangle c
    - Test $r.\mathrm{low}[i] \leq c.\mathrm{low}[i]$ and $c.\mathrm{high}[i] \leq r.\mathrm{high}[i]$, for all i
  - `r.isDisjointFrom(Rectangle c)`: true if r has no overlap with rectangle c
    - Test $r.\mathrm{high}[i] < c.\mathrm{low}[i]$ or $r.\mathrm{low}[i] > c.\mathrm{high}[i]$, for any i
    - (Not the same as `!r.contains(c)`)
  - `r.distanceFrom(Point q)`
  - Min distance from q, or 0 if q lies within r









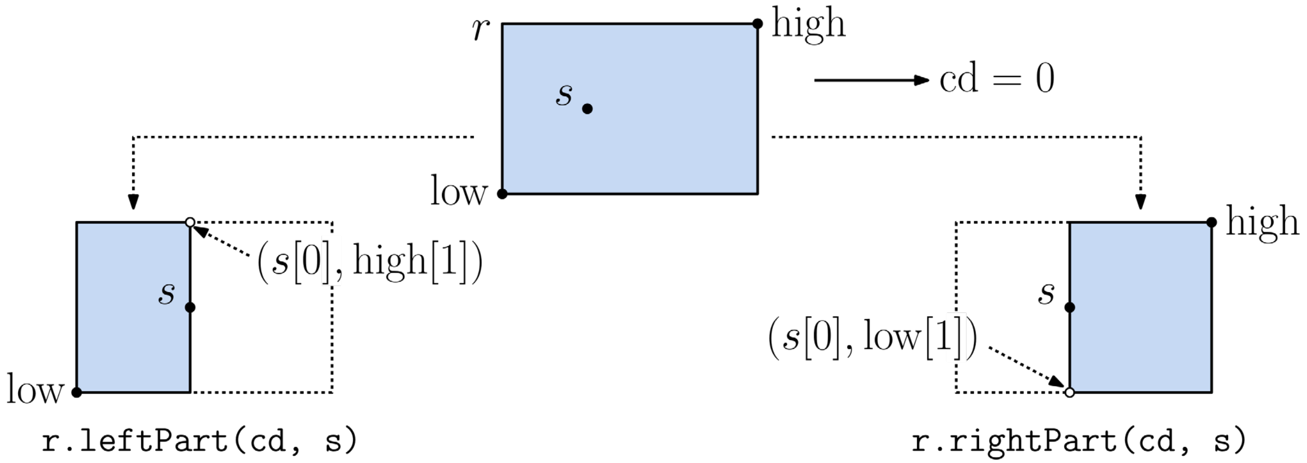
Orthogonal Range Queries

A Rectangle Class

- For manipulating kd-tree cells:
  - Given a rectangle r, a point s lying within r, and a cutting dimension cd
    - `r.leftPart(int cd, Point s)`: Portion of r left of (below) $s[cd]$

      `low` is unchanged; `high` is same except `high[cd] = s[cd]`
    - `r.rightPart(int cd, Point s)`: Portion of r right of (above) $s[cd]$

      `high` is unchanged; `low` is the same except `low[cd] = s[cd]`



Orthogonal Range Queries

A Rectangle Class

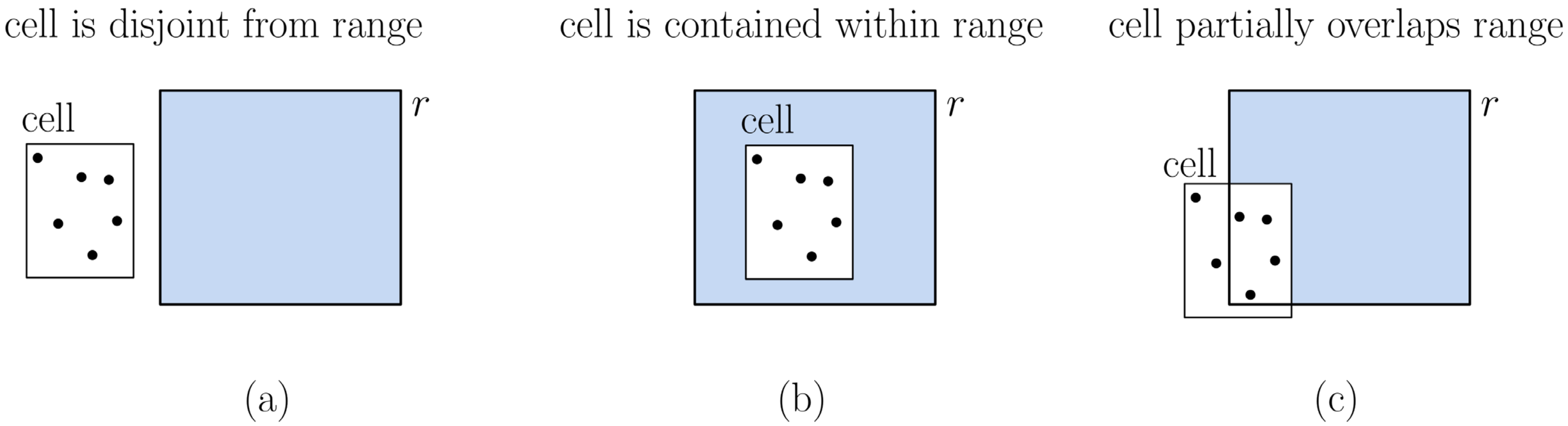
- Basic signature of the Rectangle class:

```
public class Rectangle {
   Point low;                                // lower left corner
   Point high;                               // upper right corner

   public Rectangle(Point low, Point high)   // constructor
   public boolean contains(Point q)          // do we contain q?
   public boolean contains(Rectangle c)      // do we contain rectangle c?
   public boolean isDisjointFrom(Rectangle c) // disjoint from rectangle c?
   public float distanceTo(Point q)          // min distance to point q
   public Rectangle leftPart(int cd, Point s) // left part from s
   public Rectangle rightPart(int cd, Point s) // right part from s
}
```

Orthogonal Range Queries

Answering Queries

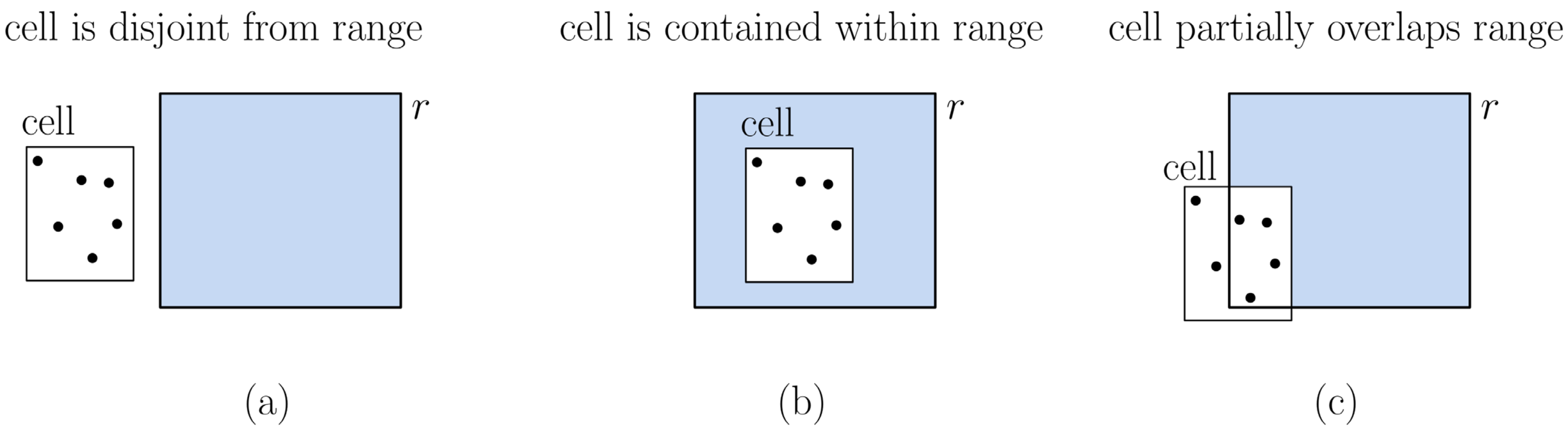
- Intuition:
  - Each node of the kd-tree is associated with a cell, a rectangular region of space based on the intersection of the cuts of its ancestors
  - As a starting point, assume that there is a bounding box, the root's cell
  - Use the cell-range relationship to avoid visiting subtrees whenever possible



Orthogonal Range Queries

Answering Queries

- Cases:
  - Cell disjoint from range: No overlap with range. Return 0
  - Cell contained in range: All the points in this subtree lie in the range. Count them all. (Assume each node p stores its subtree size, p.size)
  - Cell partially overlaps range:
    - Check whether the node's point lies in the range - if so count it
    - Recurse on both children



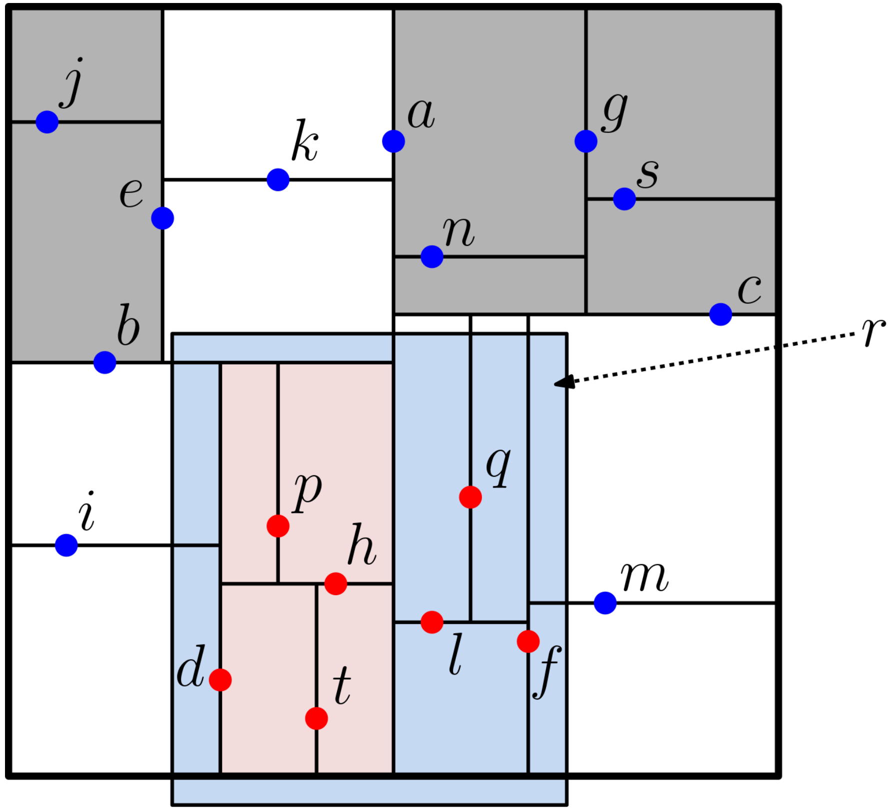
Orthogonal Range Queries

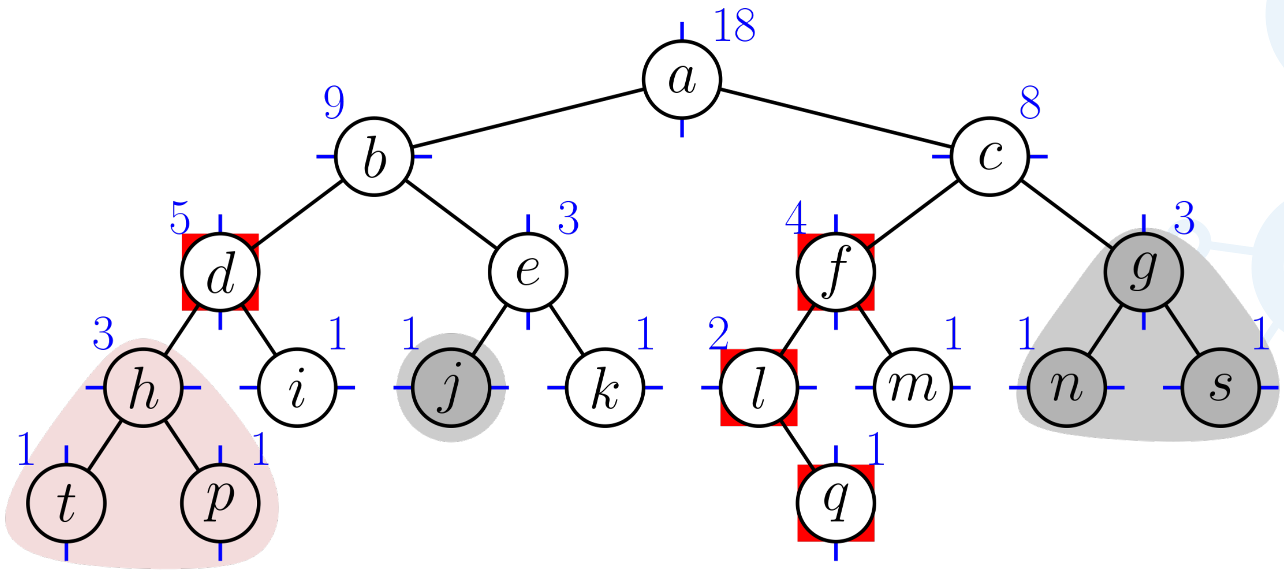
Answering Queries

```
int rangeCount(Rectangle r, KDNode p, Rectangle cell) {
    if (p == null) return 0;            // empty subtree
    else if (r.isDisjointFrom(cell))    // no overlap?
        return 0;
    else if (r.contains(cell))          // range contains our entire cell?
        return p.size;                  // …include all points in the count
    else {                              // partial overlap?
        int count = 0;
        if (r.contains(p.point))        // check this point
            count++;
                                        // apply recursively to children
        count += rangeCount(r, p.left,  cell.leftPart(p.cutDim, p.point));
        count += rangeCount(r, p.right, cell.rightPart(p.cutDim, p.point));
        return count;
    }
}
```

Orthogonal Range Queries

Example





Orthogonal Range Queries

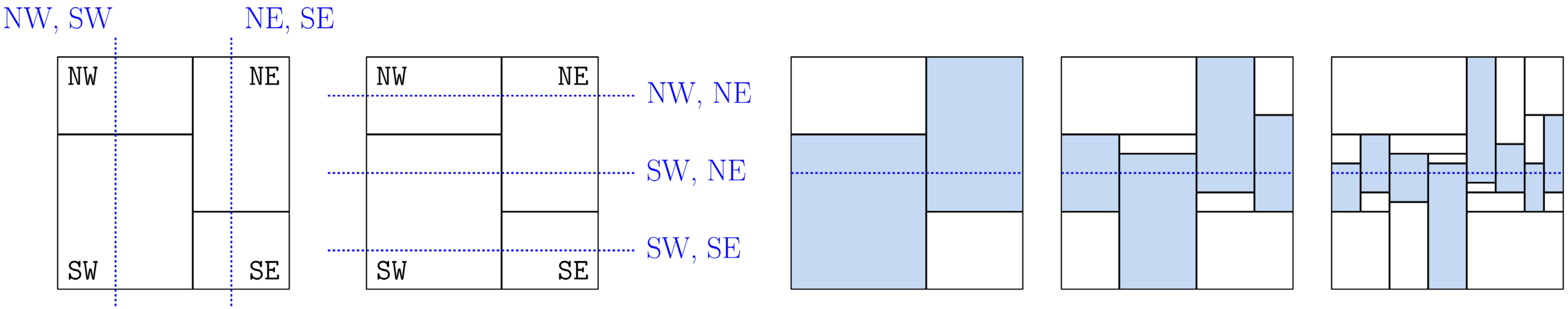
Analysis

- Theorem: Given a balanced kd-tree with n points in 2D, range counting queries can be answered in $O(\sqrt{n})$ time.
- Terminology:
  - A node p is stabbed by a line if the line intersects the interior of p’s cell
  - Observe that if a node is not stabbed by any of the four lines bounding the range, we will never recurse into this node
- Lemma: Given a balanced kd-tree with n points in 2D, the number of nodes stabbed by any axis-parallel line is $O(\sqrt{n})$.
- The above theorem follows directly from this.

Orthogonal Range Queries

Analysis

- Useful observation:
  - In 2D, if an axis-parallel line stabs a node u, then it stabs at most 2 of u's grandchildren
  - Therefore, the number of nodes stabbed at level $2i$ is at most 2^i



Orthogonal Range Queries

Analysis

- Lemma: Given a balanced kd-tree with n points in 2D, the number of nodes stabbed by any axis-parallel line is $O(\sqrt{n})$.
- Proof:
  - Let $h \approx \lg n$ be the tree height. Let l be an axis parallel line
  - If l stabs a node u, then it stabs at most 2 of u's grandchildren
  - For every two levels of the tree, the number of stabbed nodes at most doubles
  - Total number of stabbed nodes is roughly:

$$\sum_{i=0}^{h/2} 2^i \approx 2^{h/2} = \left(2^h\right)^{1/2} \approx \left(2^{\lg n}\right)^{1/2} = (n)^{1/2} = \sqrt{n}$$

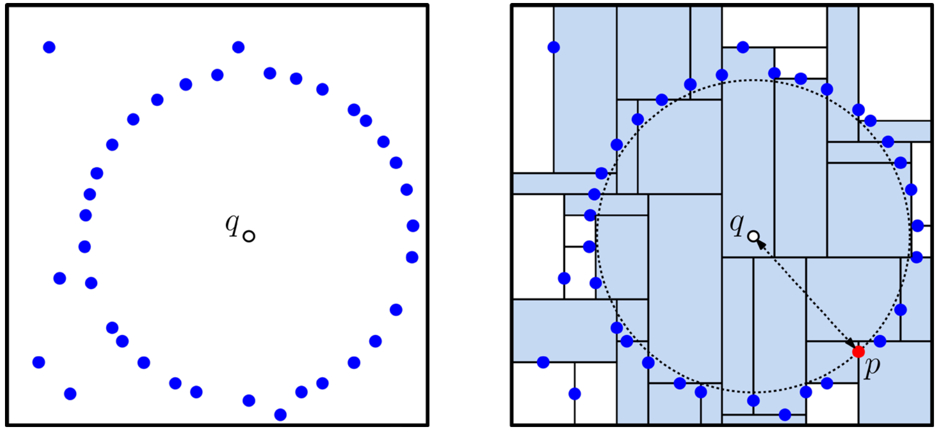
- Proof of Theorem:
  - Each of the 4 sides of the range stabs $O(\sqrt{n})$ nodes. Total time $\sim O(4\sqrt{n}) = O(\sqrt{n})$

Nearest-Neighbor Searching

- Nearest Neighbors
  - Given a kd-tree and a query point q, compute the closest point in the kd-tree to q
  - We assume that distances are measured using the Euclidean metric:

$$\mathrm{dist}(p, q) = \sqrt{(p_1 - q_1)^2 + \cdots + (p_d - q_d)^2}$$

  - Unfortunately, worst case is $O(n)$, which happens if almost all points at same distance. In practice, much better



Nearest-Neighbor Searching

- Overview:
  - For simplicity, we will compute just the distance to the nearest neighbor
    - Computing the actual point is a simple extension
  - Search operates recursively, starting from the root
  - Keep track of the minimum distance to the query seen so far - `bestDist`
  - Minimize the number of nodes visited:
    - Visit the subtree (left or right) that is closer to the query point first
    - Don’t visit the other child if it cannot possibly contribute a closer point

Nearest-Neighbor Searching

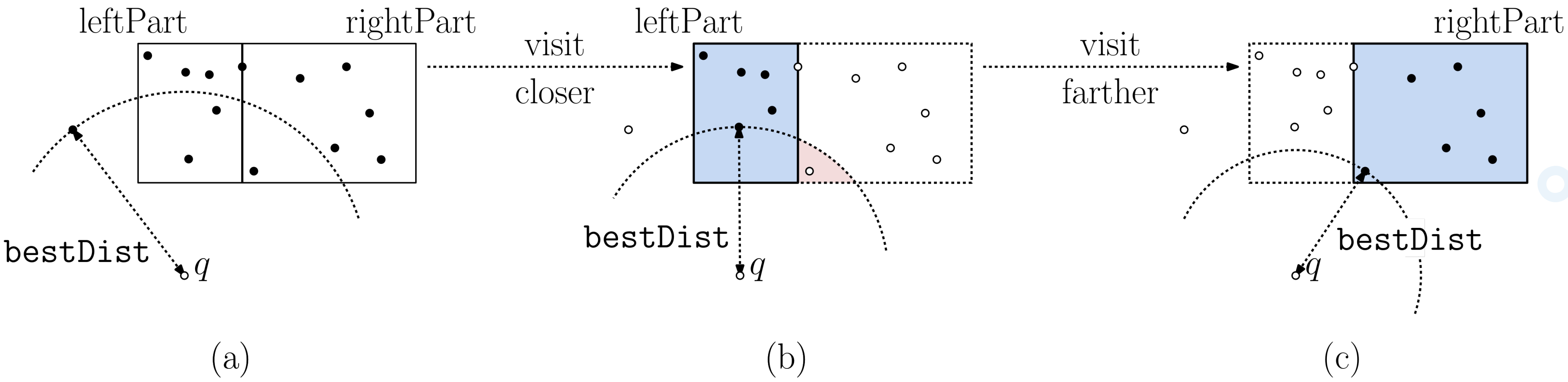
Answering Queries

- float nearNeighbor(Point q, Node p, Rectangle cell, float bestDist)
  - If p is `null` - return `bestDist` (empty subtree, no change in best)
  - Else:
    - Compute `dist(q, p.point)` and update `bestDist` if this is smaller
    - Compute child cells, `leftPart` and `rightPart`
    - Determine which child is closer to the query point (which side is q w.r.t. splitter)
    - Recursively visit the closer child - Update `bestDist`
    - Visit the farther child only if it is sufficiently close - Update `bestDist`
    - Return `bestDist`

Nearest-Neighbor Searching

Answering Queries

- float nearNeighbor(Point q, Node p, Rectangle cell, float bestDist)

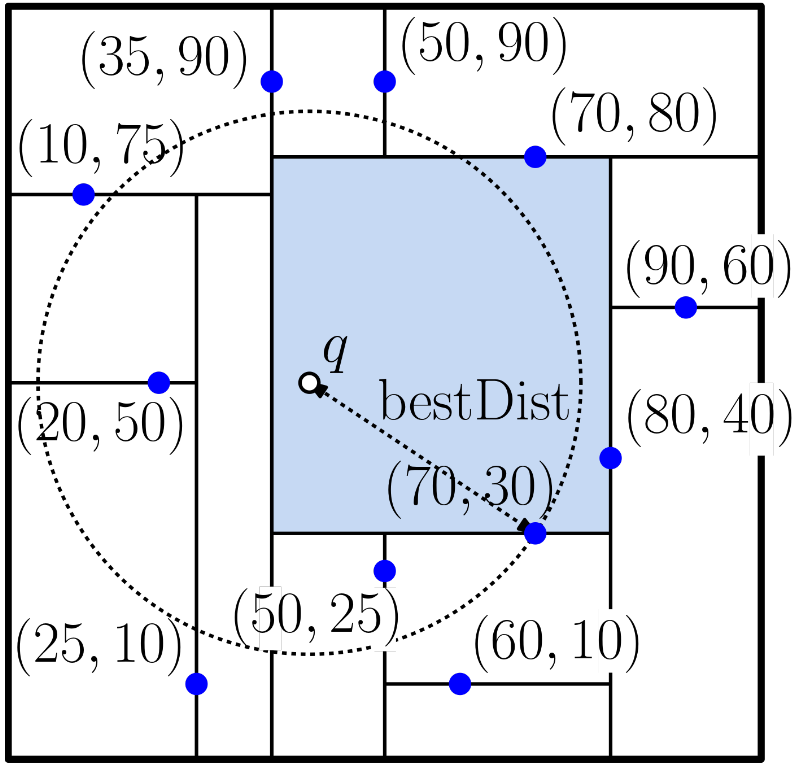


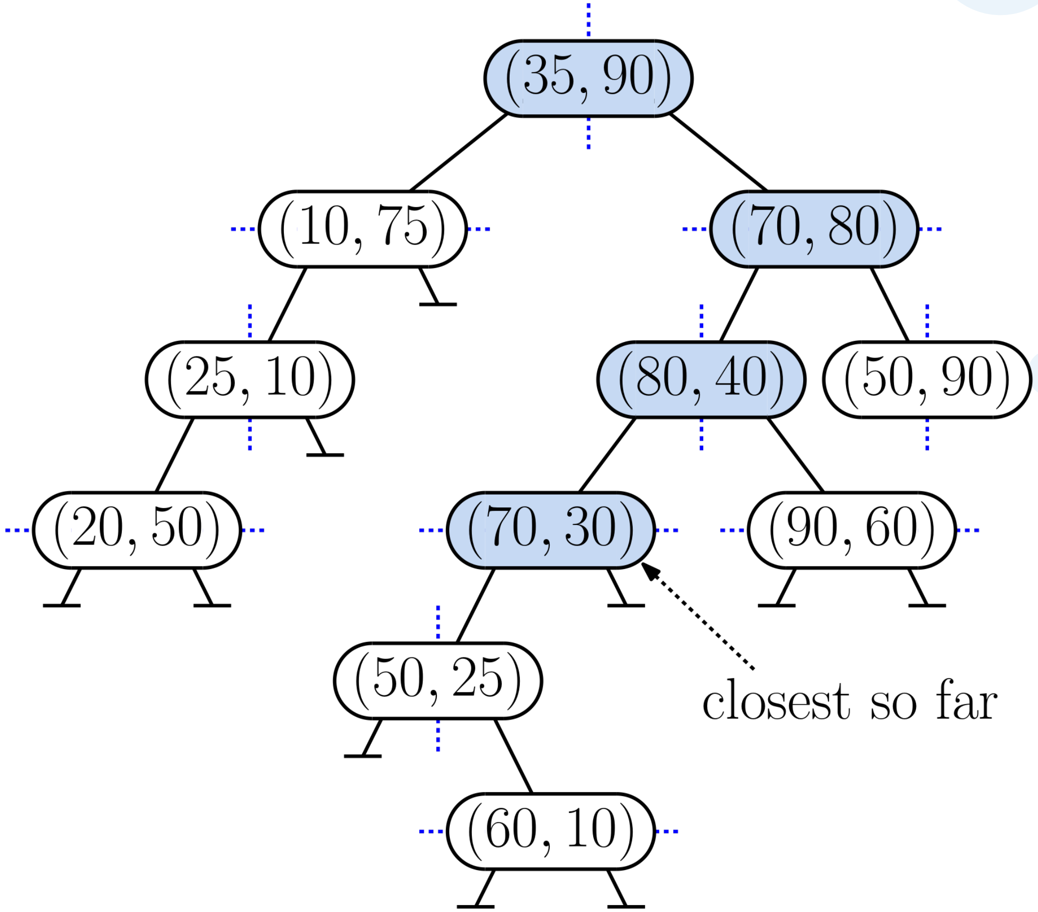
Nearest-Neighbor Searching

```
float nearNeighbor(Point q, KDNode p, Rectangle cell, float bestDist) {
    if (p != null) {
        float thisDist = q.distanceTo(p.point);             // distance to p's point
        bestDist = Math.min(thisDist,  bestDist);           // keep smaller distance

        int cd = p.cutDim;                                  // cutting dimension
        Rectangle leftCell  = cell.leftPart(cd, p.point);   // left child's cell
        Rectangle rightCell = cell.rightPart(cd, p.point);  // right child's cell

        if (q[cd] < p.point[cd]) {                          // q is closer to left
            bestDist = nearNeighbor(q, p.left, leftCell, bestDist);
            if (rightCell.distanceTo(q) < bestDist) {       // worth visiting right?
                bestDist = nearNeighbor(q, p.right, rightCell, bestDist);
            }
        } else {                                            // q is closer to right
            /* … left-right symmetrical …  */
        }
    }
    return bestDist;
}
```

Nearest-Neighbor Searching

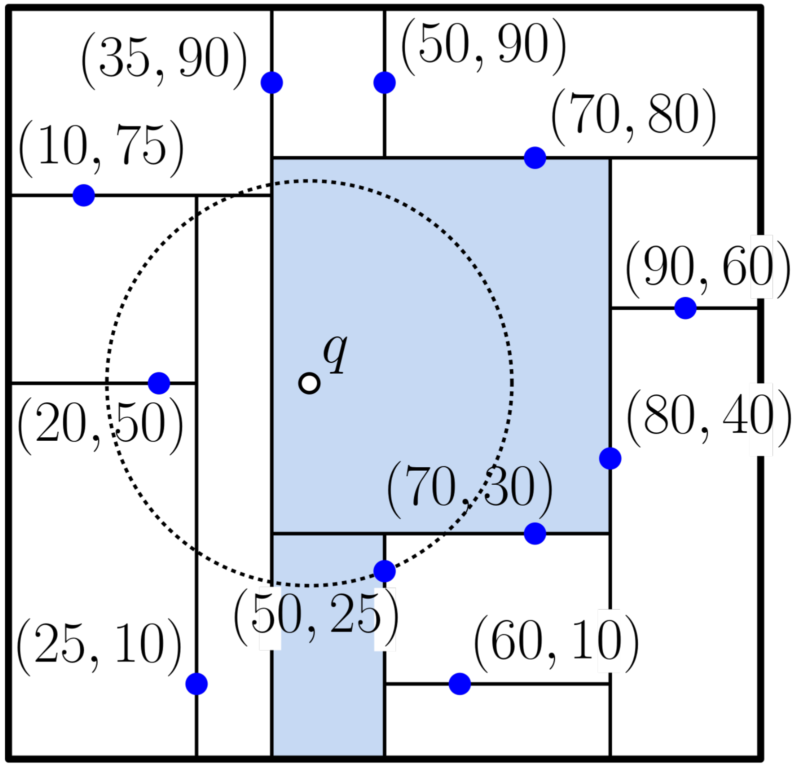
Example

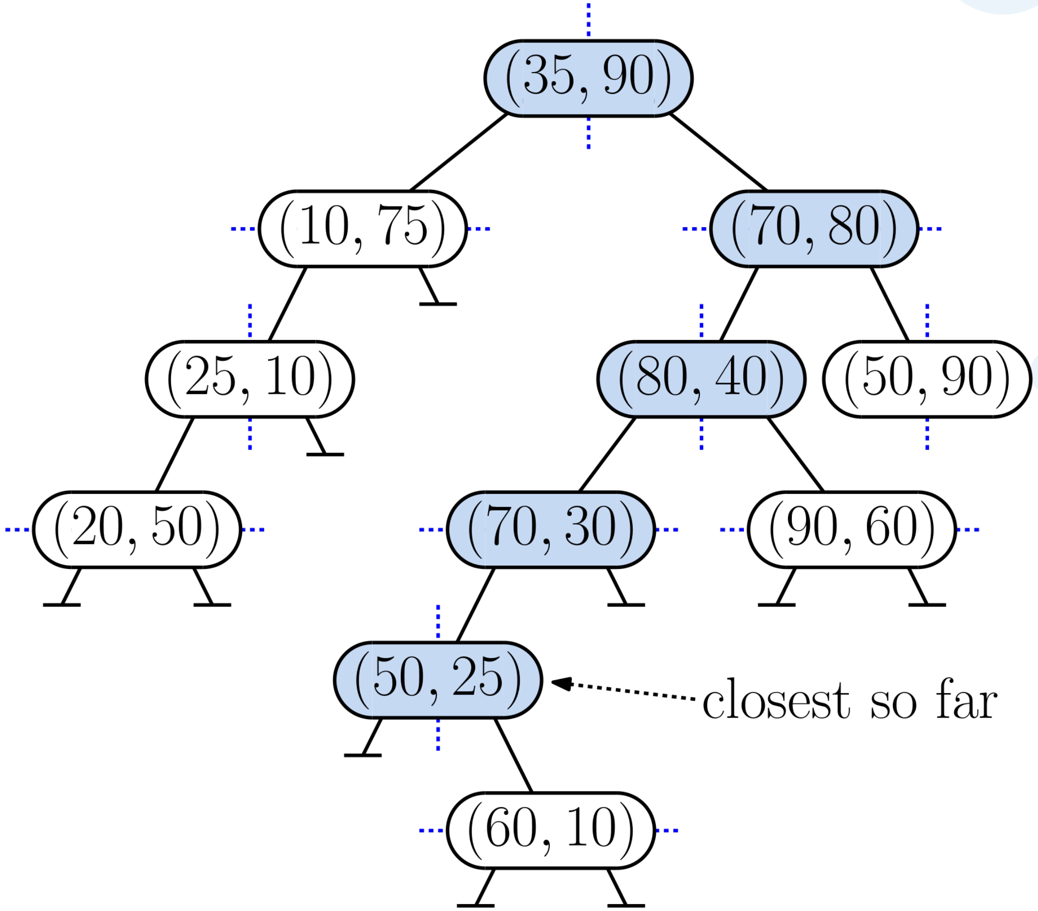




Nearest-Neighbor Searching

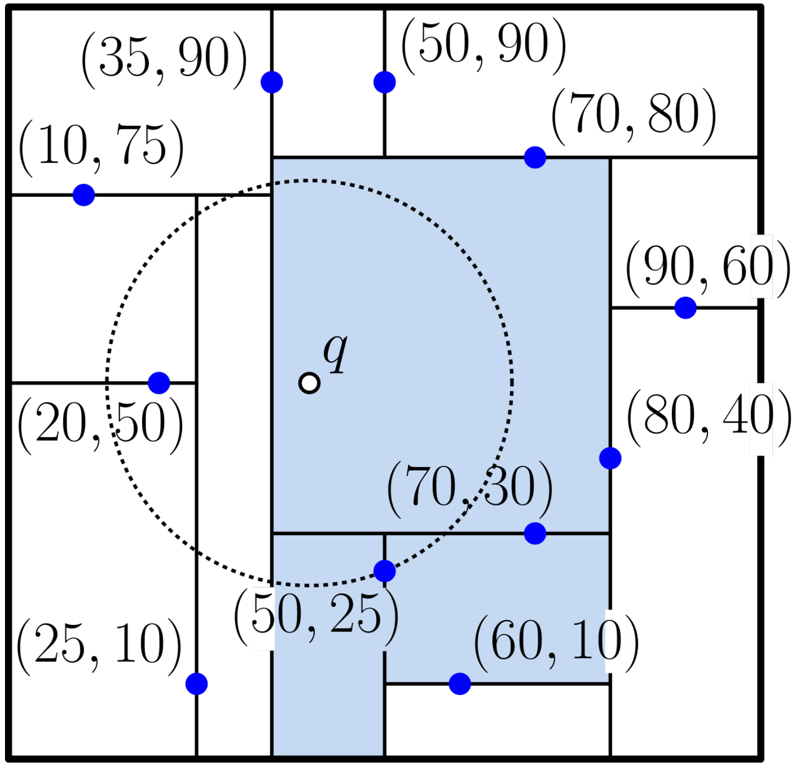
Example

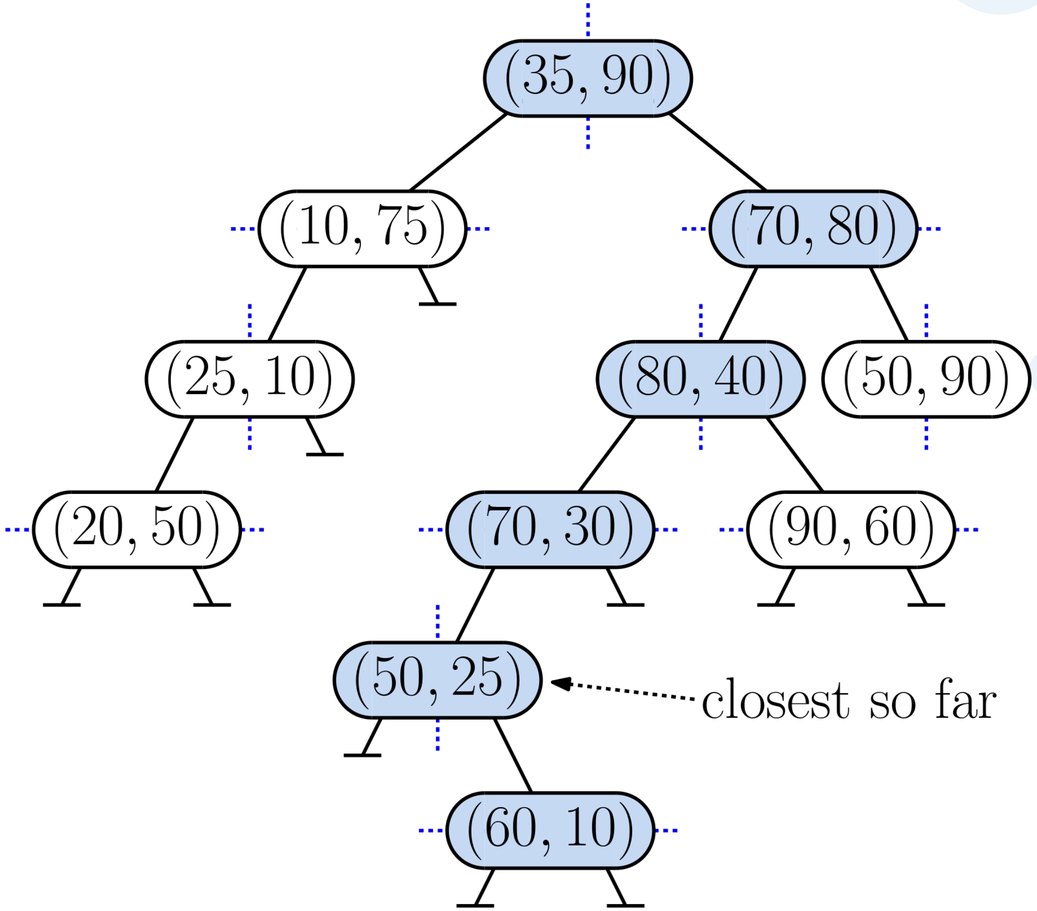




Nearest-Neighbor Searching

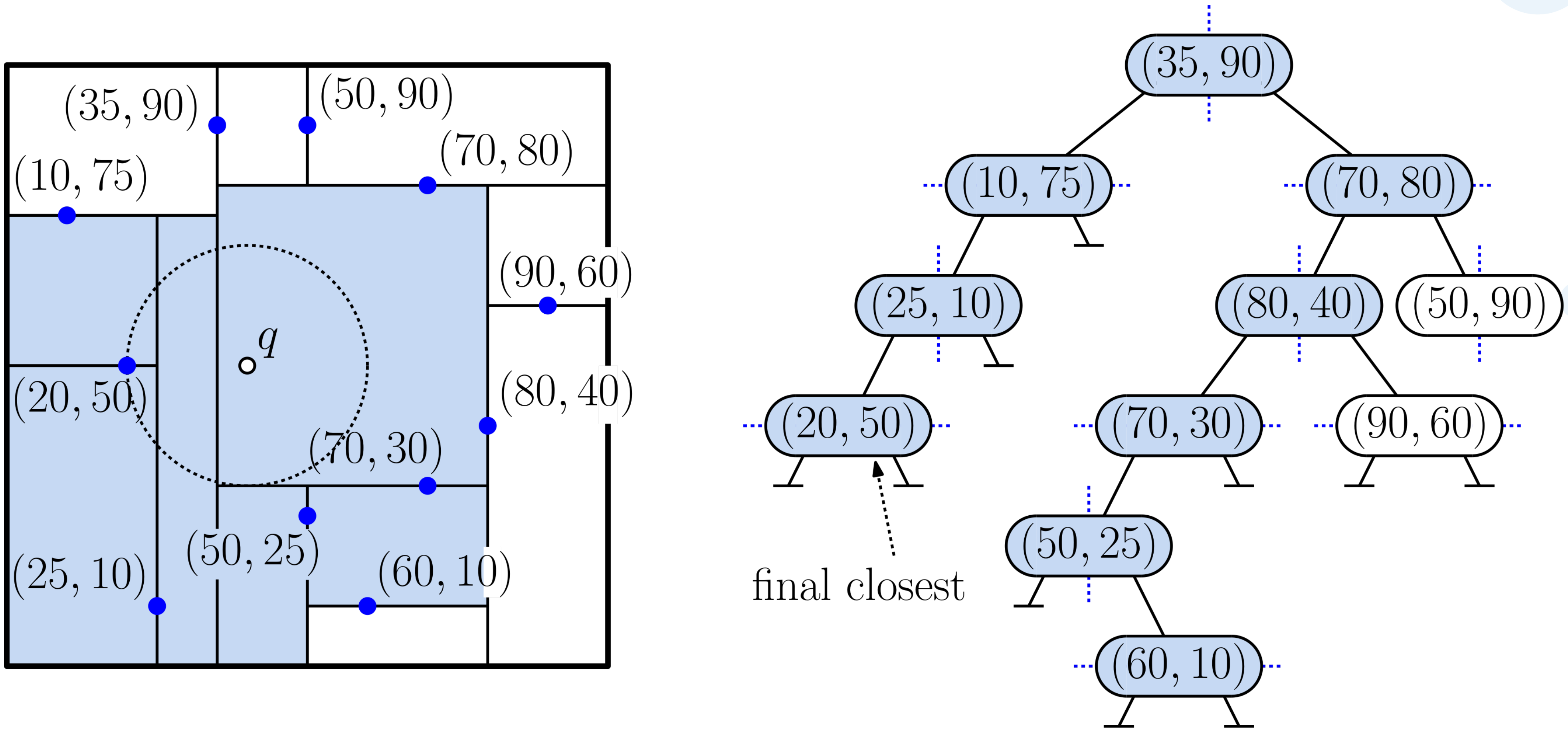
Example





Nearest-Neighbor Searching

Example



Summary

- Answering Queries with kd-trees
  - Principles:
    - Use recursion to visit subtrees
    - Maintain intermediate results
    - Avoid visiting subtrees whenever possible
  - Orthogonal range (counting) queries
  - Nearest-neighbor queries