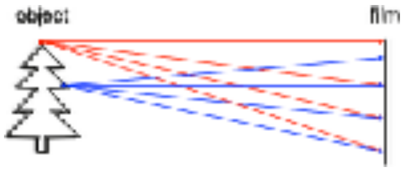
Geometric Transformations

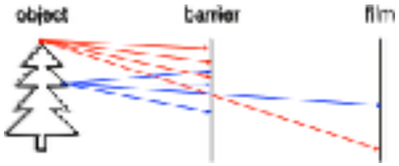
Let's design a camera



- Idea 1: put a piece of film in front of an object
- Do we get a reasonable image?

Slide by Steve Seitz

Pinhole camera



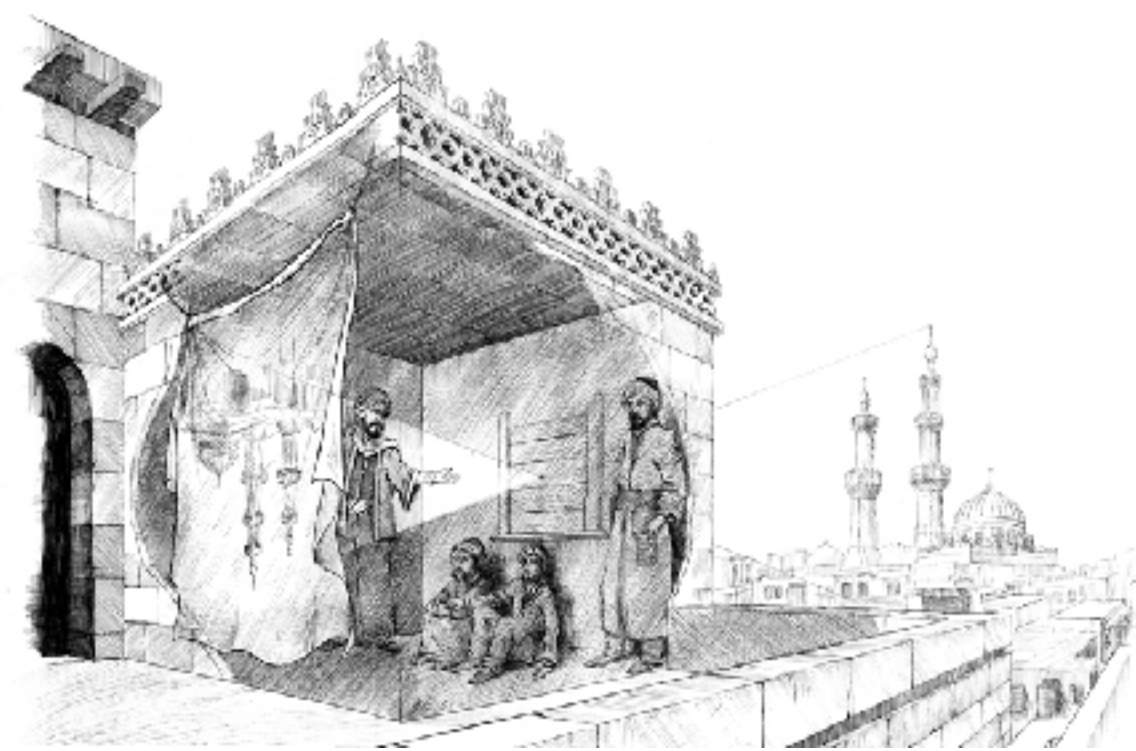
- Add a barrier to block off most of the rays
  - This reduces blurring
  - The opening is known as the **aperture**

Slide by Steve Seitz

Camera Obscura

First Idea:Mo-Ti , China (470 BC to 390 BC)

First built: Ibn Al-Haytham or Alhazen, Iraq/Egypt (965 to 1039AD)



Camera Obscura

illum in tabula per radios Solis, quàm in cœlo contin-
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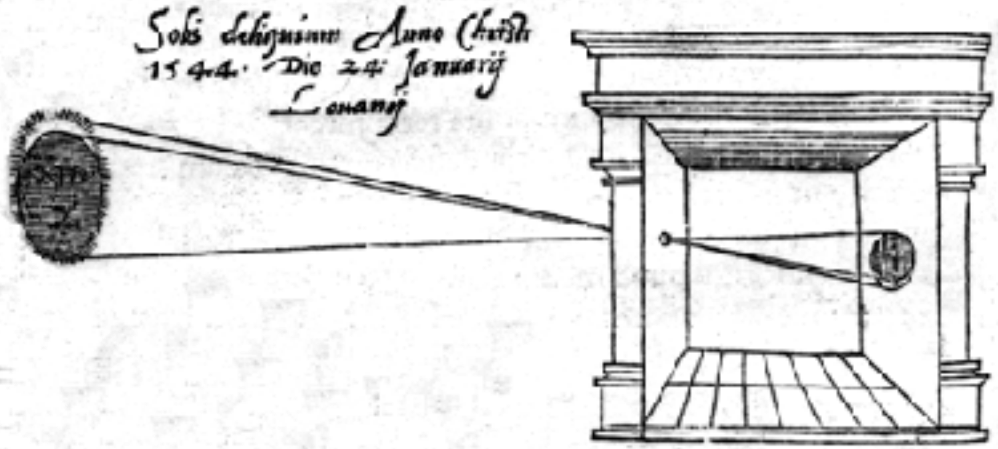


Sic nos exactè Anno .1544. Louanii eclipſim Solis
obſeruauimus, inuenimusq; deficere paulò plus q̃ dex-

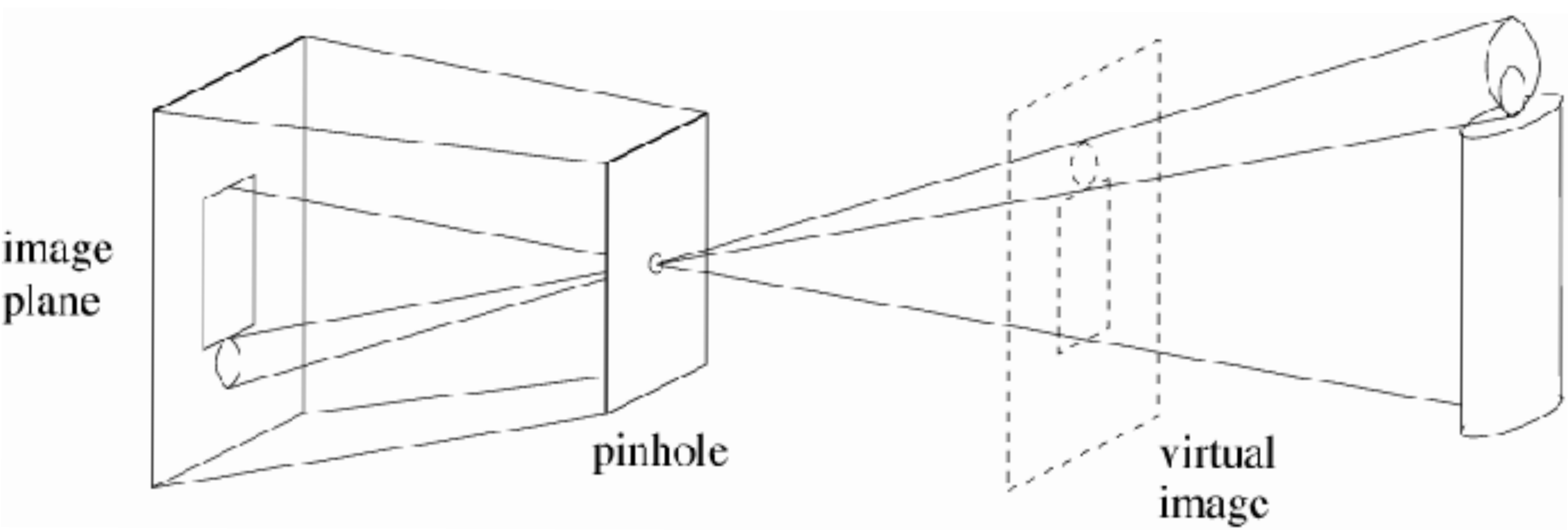
"When images of illuminated objects ... penetrate through a small hole into a very dark room ... you will see [on the opposite wall] these objects in their proper form and color, reduced in size ... in a reversed position, owing to the intersection of the rays".

Da Vinci

http://www.acmi.net.au/AIC/CAMERA_OBSCURA.html (Russell Naughton)

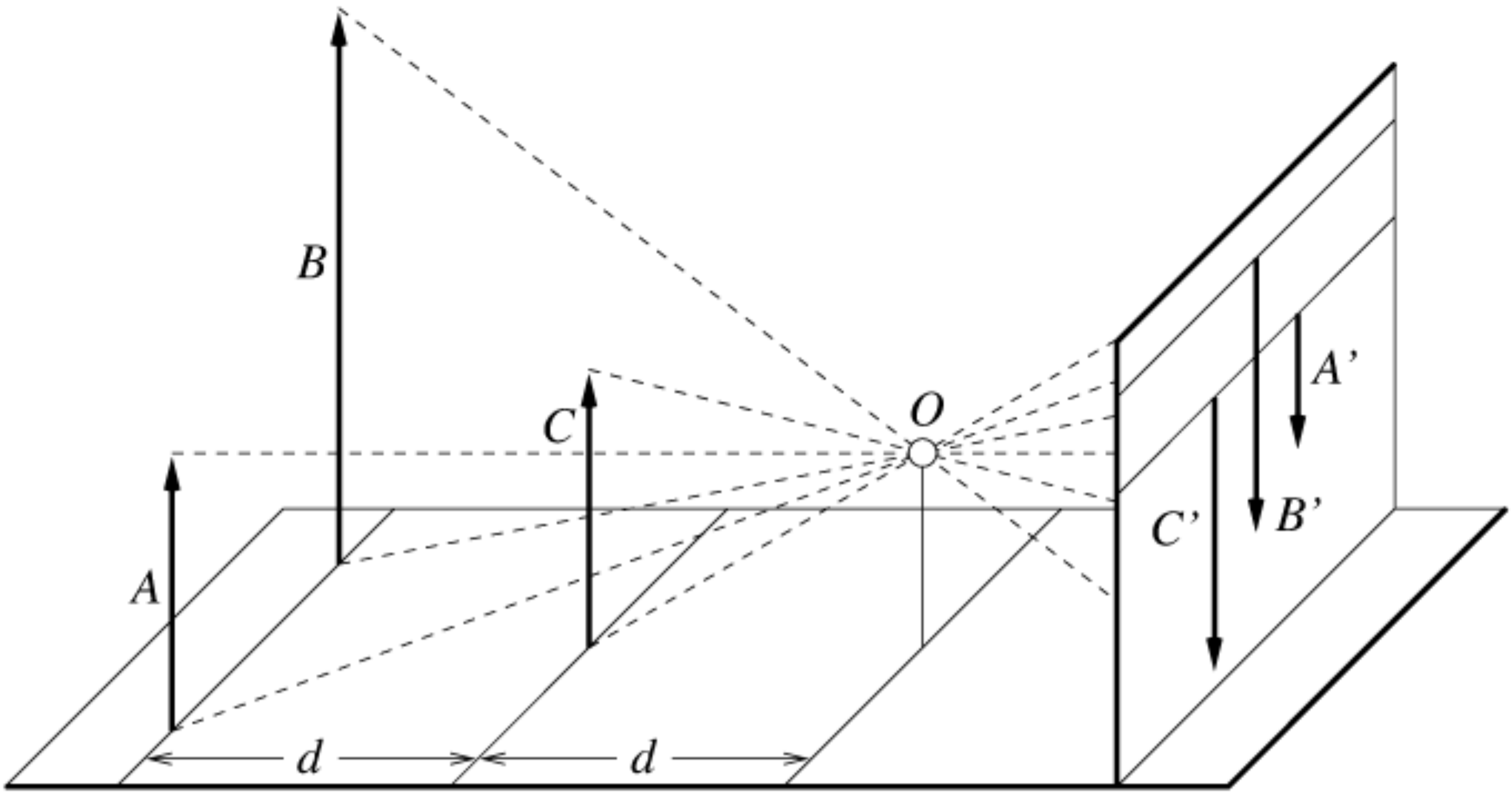
Pinhole cameras

- Abstract camera model - box with a small hole in it
- Pinhole cameras work in practice



(Forsyth & Ponce)

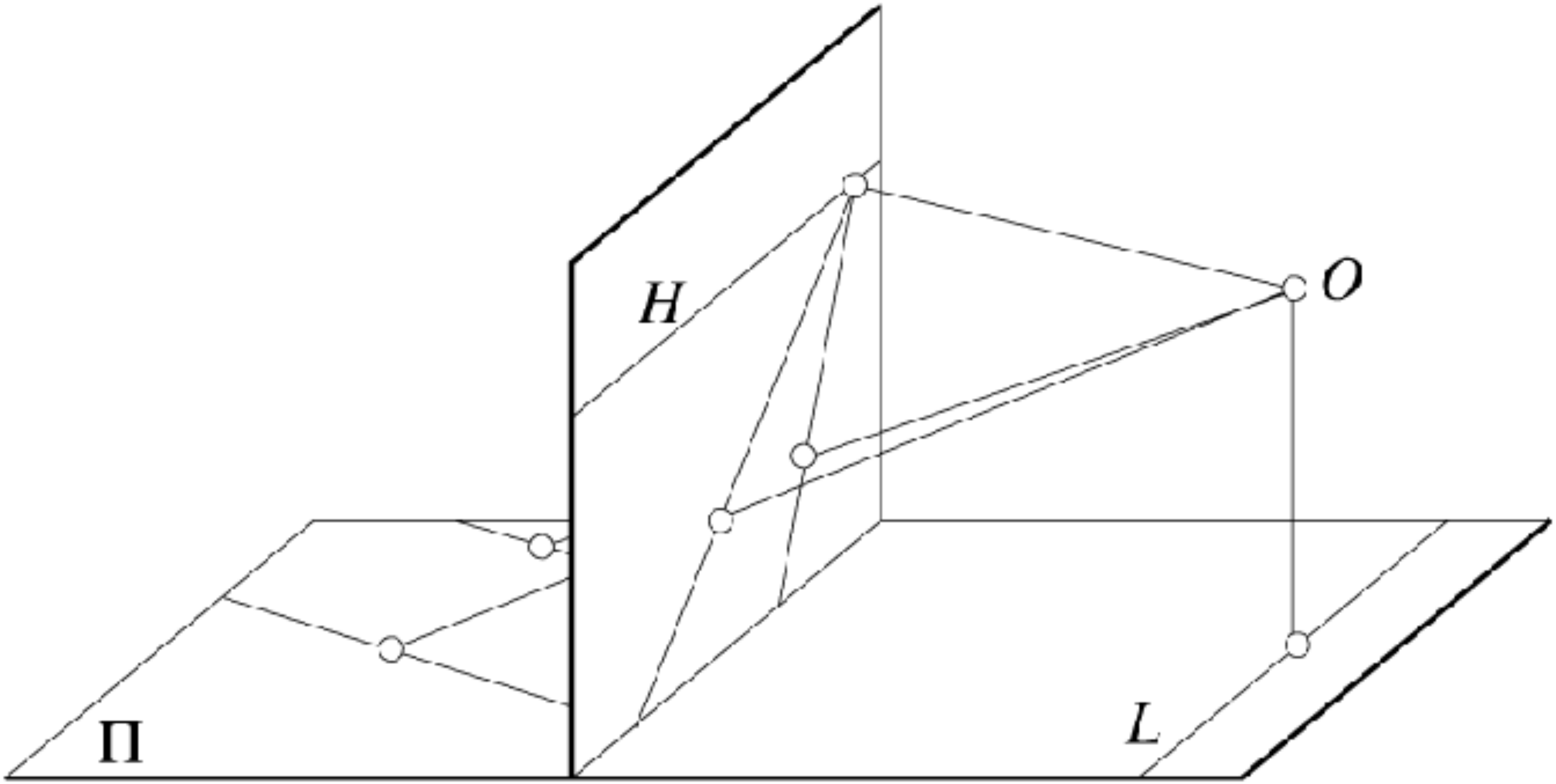
Distant objects are smaller



(Forsyth & Ponce)

Parallel lines meet

Common to draw image plane *in front* of the focal point.
Moving the image plane merely scales the image.



(Forsyth & Ponce)

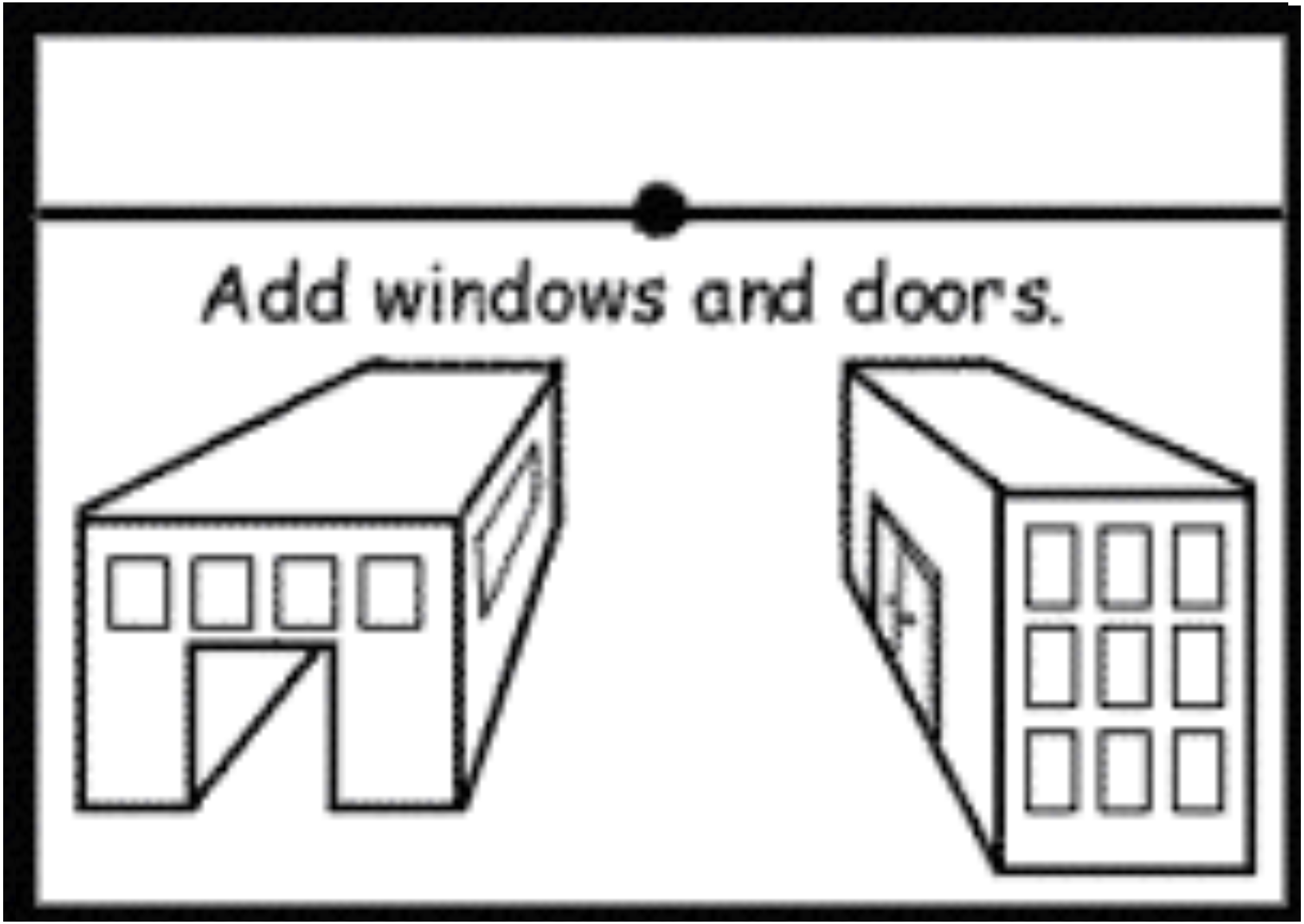
Vanishing points

- Each set of parallel lines meets at a different point
  - The *vanishing point* for this direction
- Sets of parallel lines on the same plane lead to *collinear* vanishing points.
  - The line is called the *horizon* for that plane

Properties of Projection

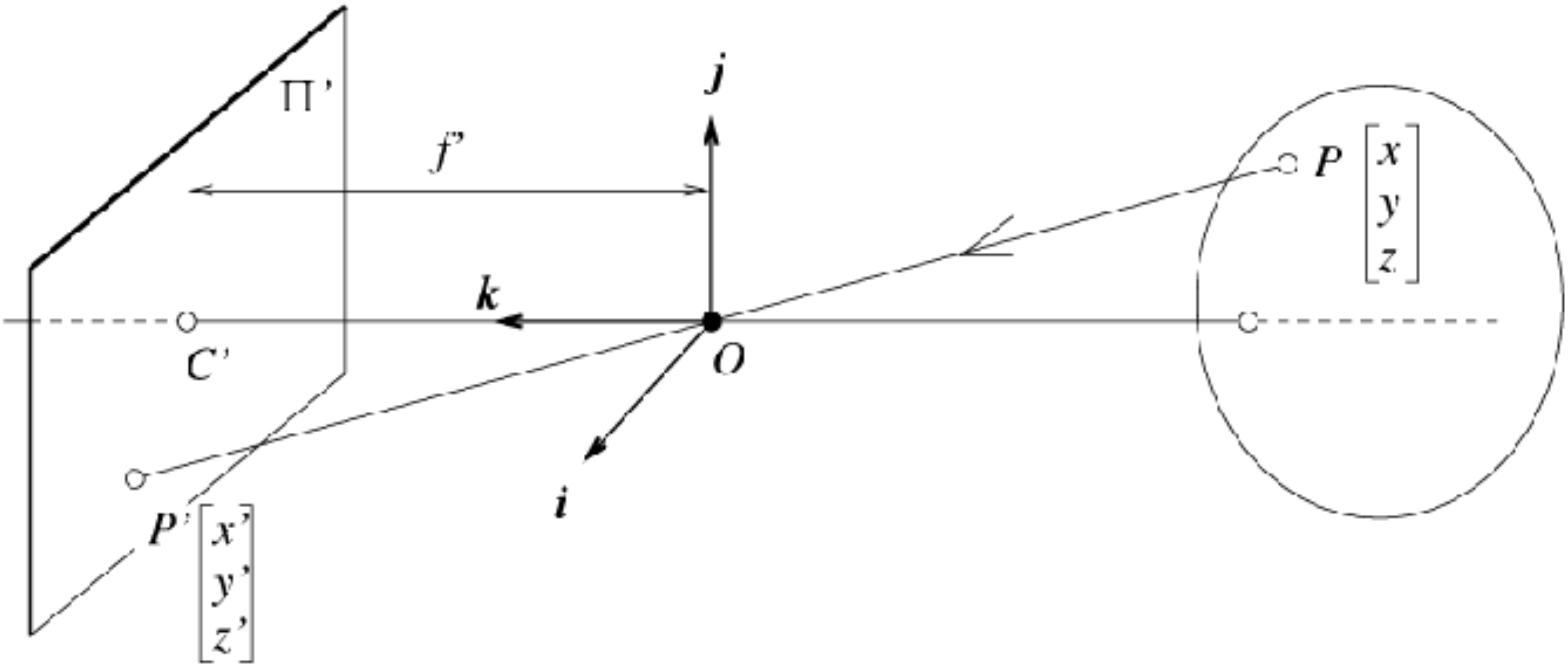
- Points project to points
- Lines project to lines
- Planes project to the whole image or a half image
- Angles are not preserved
- Degenerate cases
  - Line through focal point projects to a point.
  - Plane through focal point projects to line
  - Plane perpendicular to image plane projects to part of the image (with horizon).

Take out paper and pencil



http://www.sanford-artedventures.com/create/tech_1pt_perspective.html

The equation of projection



(Forsyth & Ponce)

The equation of projection

- Cartesian coordinates:
  - We have, by similar triangles, that

$$x' = f'\frac{x}{z}$$

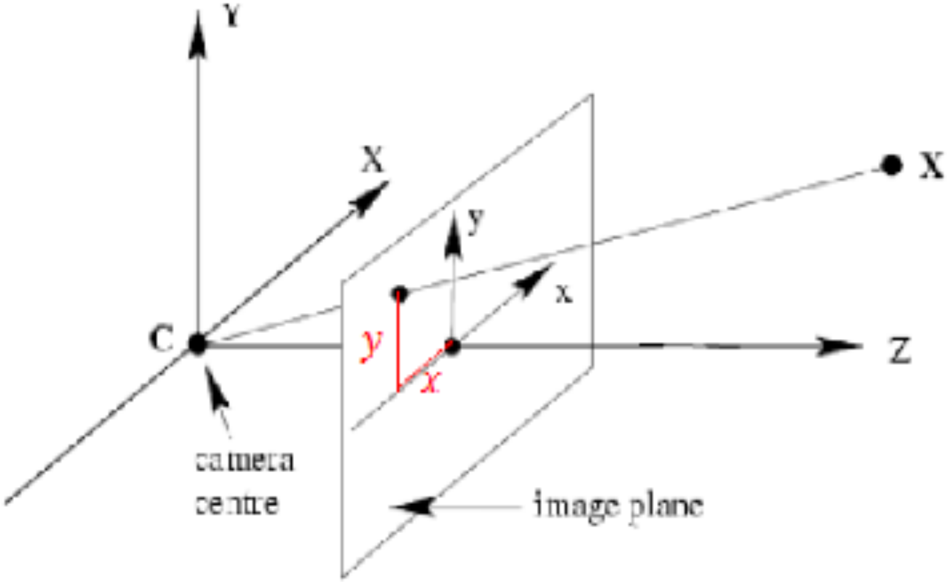
$$y' = f'\frac{y}{z}$$

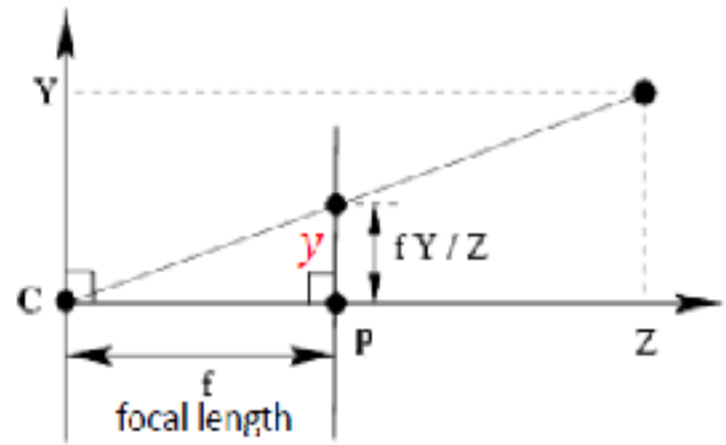
$$(x, y, z) \rightarrow (f'\frac{x}{z}, f'\frac{y}{z}, f')$$

  - Ignore the third coordinate, and get

$$(x, y, z) \rightarrow (f'\frac{x}{z}, f'\frac{y}{z})$$

Pinhole camera model – in maths

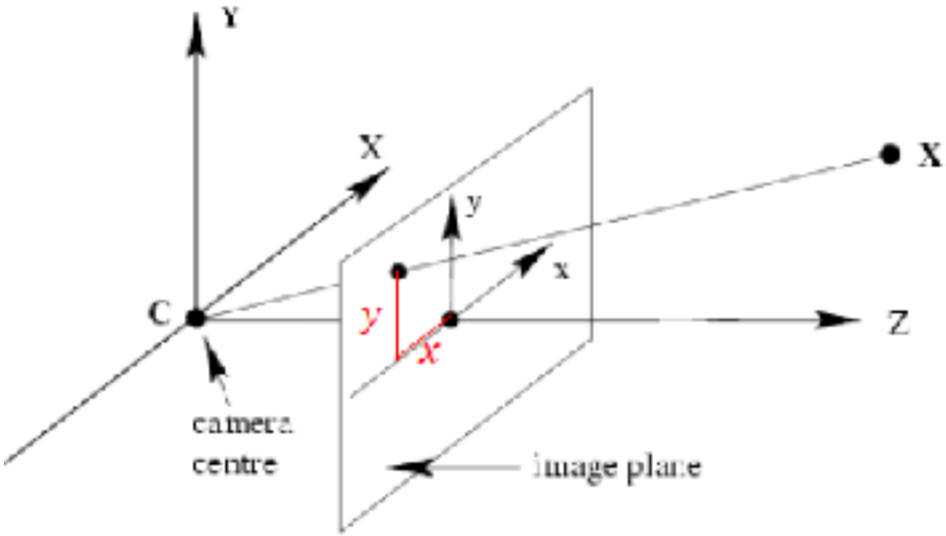


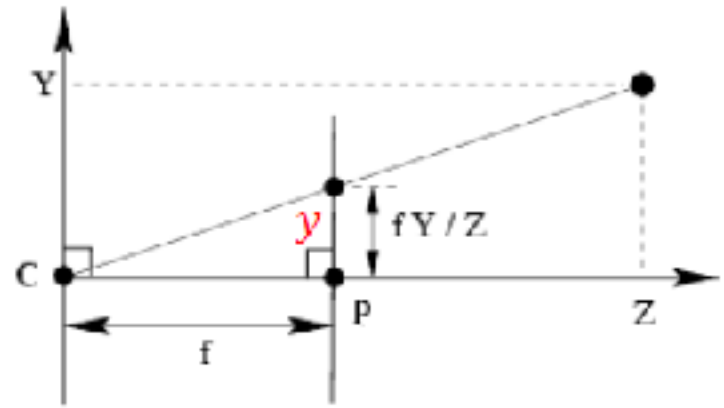


- Similar trinagles: $\frac{y}{f} = \frac{Y}{Z}$
- That gives: $y = f\frac{Y}{Z}$ and $x = f\frac{X}{Z}$

That gives: $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$

Pinhole camera model – in maths





That gives:

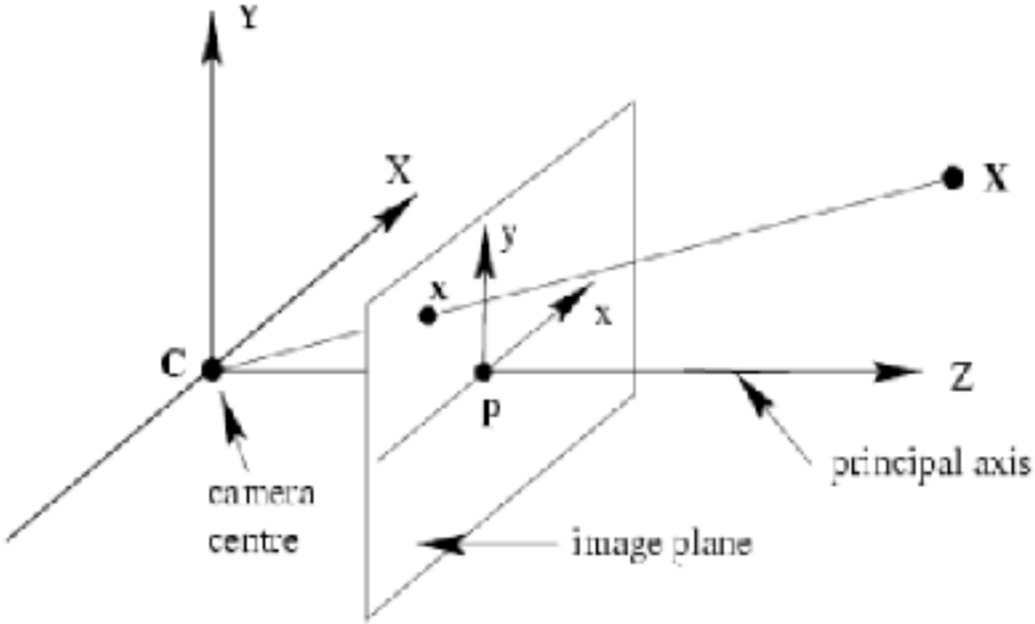
$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Calibration matrix } K} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

In short $x = K\tilde{X}$ (here $\tilde{X}$ means inhomogeneous coordinates)

Intrinsic Camera Calibration means we know K (we do that later)

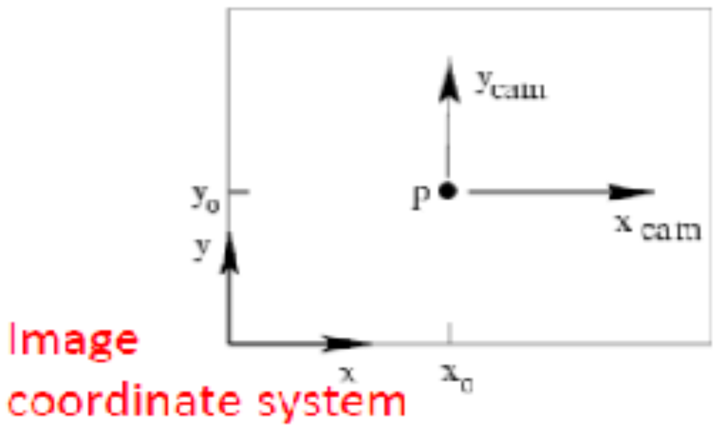
We can go from image points into the 3D world: $\tilde{X} = K^{-1} x$

Pinhole camera - definitions



- Principal axis: line from the camera center perpendicular to the image plane
- Normalized (camera) coordinate system: camera center is at the origin and the principal axis is the z-axis
- Principal point (p): point where principal axis intersects the image plane (origin of normalized coordinate system)

Principal Point



Principal point (p_x, p_y)

- Camera coordinate system: origin is at the principal point
- Image coordinate system: origin is in the corner
  In practice: principal point in center of the image

Adding principal point into $\boldsymbol{K}$

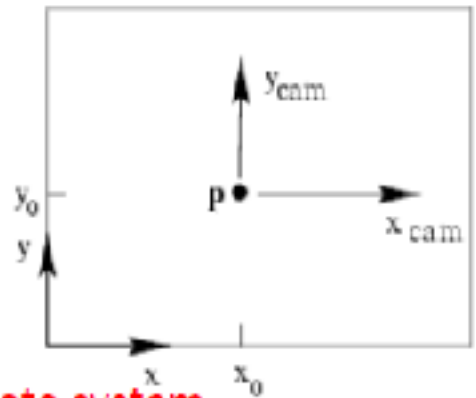


Image coordinate system

Principal point (p_x, p_y)

Projection with principal point : $y = f\frac{Y}{Z} + p_y = \frac{fY+Zp_y}{Z}$ and $x = f\frac{X}{Z} + p_x = \frac{fX+Zp_x}{Z}$

That gives: $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$

Adding principal point into $\boldsymbol{K}$

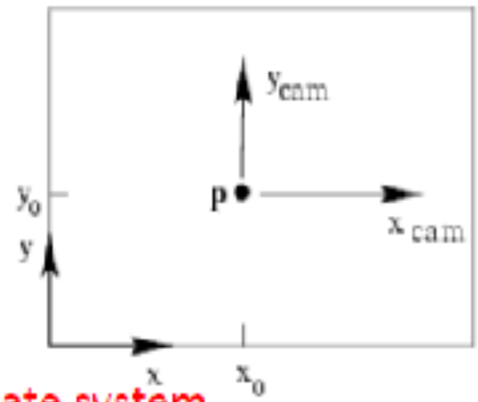


Image coordinate system

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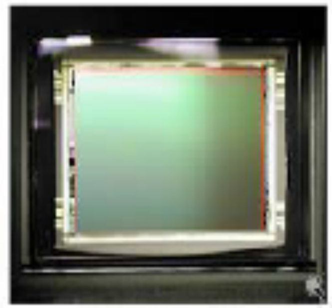
Intrinsic matrix, K

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pixel Size and Shape



- m_x pixels per unit (m,mm,inch,...) in horizontal direction
- m_y pixels per unit (m,mm,inch,...) in vertical direction
- s' skew of a pixel
- *In practice (close to): m=1 s = 0*

That gives: $$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} m_x & s' & 0 \\ 0 & m_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Simplified to: $$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & s & p_x \\ 0 & mf & p_y \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Final calibration matrix } \boldsymbol{K}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

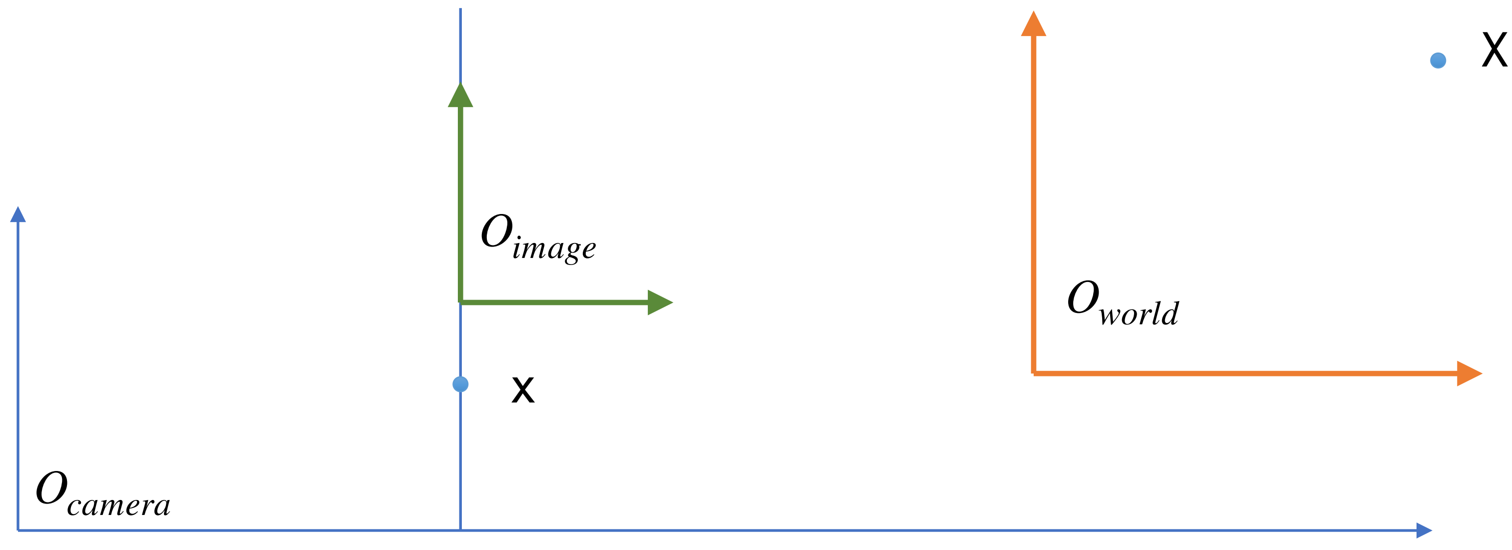
f now in units of pixels

Camera intrinsic parameters - Summary

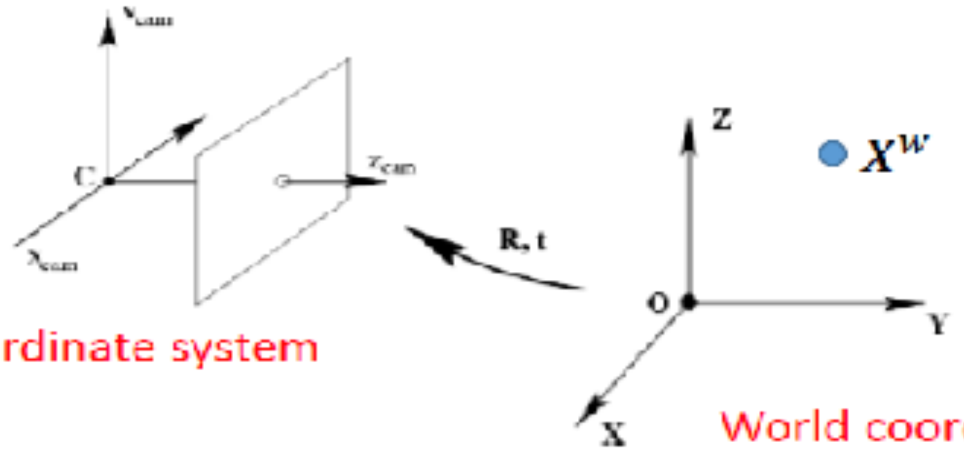
- Intrinsic parameters
  - Principal point coordinates (p_x, p_y)
  - Focal length f
  - Pixel magnification factors m
  - Skew (non-rectangular pixels) s

$$\boldsymbol{K} = \begin{bmatrix} f & s & p_x \\ 0 & mf & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

Three different coordinate systems



Putting the camera into the world



camera coordinate system

World coordinate system

Given a 3D homogenous point X^W in world coordinate system

1) Translate from world to camera coordinate system:

$$\tilde{X}^{c'} = \tilde{X}^w - \tilde{C}$$

$$\tilde{X}^{c'} = \underbrace{(I_{3\times3} \mid \quad \tilde{C})}_{3\times4 \text{ matrix}} X^W$$ where $I_{3\times3}$ is 3x3 identity matrix

2) Rotate world coordinate system into camera coordinate system

$$\tilde{X}^c = R\,(I_{3\times3} \mid -\tilde{C})\, X^W$$

3) Apply camera matrix

$$x = K\,R\,(I_{3\times3} \mid -\tilde{C})\, X$$

Camera extrinsic (or external) parameters

- Transform a point from the world coordinate to the camera's coordinate system
- Translation and rotation

$$X_c = R(X_w - C_w)$$

$$X_c = RX_w - RC_w$$

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} R & -RC_w \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} R & -RC_w \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & -C_w \\ 0 & 1 \end{bmatrix}$$

Camera extrinsic (or external) parameters

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} R & -RC_w \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} R & -RC_w \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & -C_w \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} I & -C_w \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -c_x \\ 0 & 1 & 0 & -C_y \\ 0 & 0 & 1 & -C_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Camera extrinsic (or external) parameters

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -C_x \\ 0 & 1 & 0 & -C_y \\ 0 & 0 & 1 & -C_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = R(I_{3\times3} \mid -C_w) \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = K \begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} \qquad K = \begin{bmatrix} f & s & p_x \\ 0 & mf & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

Summary

$$\begin{bmatrix} X_w \\ Y_w \\ z_w \end{bmatrix} \longrightarrow \begin{bmatrix} X_c \\ Y_c \\ z_c \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

world coordinates

camera coordinates

pixel coordinates

Camera matrix

- Camera matrix $\boldsymbol{P}$ is defined as:

$$\boldsymbol{x} = \underbrace{\boldsymbol{K}\,\boldsymbol{R}\,(\boldsymbol{I}_{3\times 3} \mid -\tilde{\boldsymbol{C}})}_{\boldsymbol{P}}\,\boldsymbol{X}$$

$\boldsymbol{P}$ (3×4) camera matrix has 11 DoF

- In short we write: $\boldsymbol{x} = \boldsymbol{P}\,\boldsymbol{X}$

Camera matrix

- Camera matrix $\boldsymbol{P}$ is defined as:

$$\boldsymbol{x} = \underbrace{\boldsymbol{K}\,\boldsymbol{R}\,(\boldsymbol{I}_{3\times 3} \mid -\tilde{\boldsymbol{C}})}_{\boldsymbol{P}\ (3\times\mathbf{4})\ \text{camera matrix has 11 DoF}}\,\boldsymbol{X}$$

- In short we write: $\boldsymbol{x} = \boldsymbol{P}\,\boldsymbol{X}$

Image of a Point

Homogeneous coordinates of a 3-D point p

$$\boldsymbol{X} = [X, Y, Z, W]^T \in \mathbb{R}^4, \quad (W = 1)$$

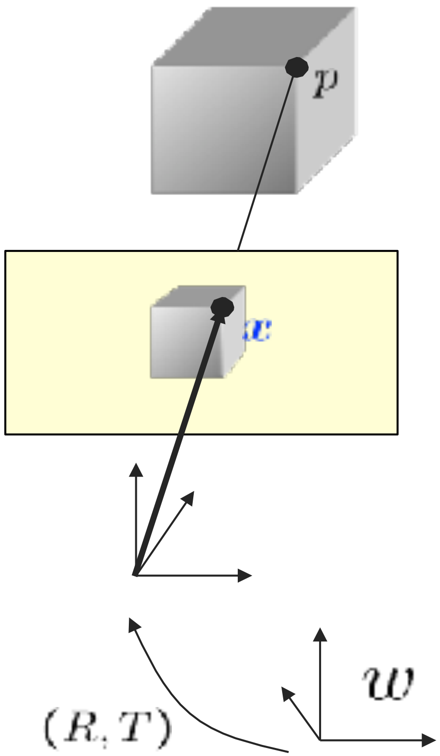
Homogeneous coordinates of its 2-D image

$$\boldsymbol{x} = [x, y, z]^T \in \mathbb{R}^3, \quad (z = 1)$$

Projection of a 3-D point to an image plane

$$\lambda x = PX$$

$$\lambda \in \mathbb{R}, \ P = [R, T] \in \mathbb{R}^{3 \times 4}$$



Camera parameters - Summary

- Camera matrix P has 11 DoF

$$x = P\,X$$

$$x = K\,R\,(I_{3\times3} \mid -\tilde{C})\,X$$

$$K = \begin{bmatrix} f & s & p_x \\ 0 & mf & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

- <u>Intrinsic parameters</u>
  - Principal point coordinates (p_x, p_y)
  - Focal length f
  - Pixel magnification factors m
  - Skew (non-rectangular pixels) s

- <u>Extrinsic parameters</u>
  - Rotation $\boldsymbol{R}$ (3DoF) and translation $\tilde{\mathbf{C}}$ (3DoF) relative to world coordinate system