Eigen values and vectors

SPECTRAL THEOREM

Theorem: If $X \in \mathbb{R}^{m \times n}$ is symmetric matrix (meaning $X^T = X$),

then, there exist real numbers $\lambda_1, \ldots, \lambda_n$ (the eigenvalues)

and orthogonal, non-zero real vectors $\phi_1, \phi_2, \ldots, \phi_n$

(the eigenvectors) such that for each $i = 1, 2, \ldots, n$:

$$X\phi_i = \lambda_i \phi_i$$

EXAMPLE

$$A = \begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix}$$

From spectral theorem:

$$A\phi = \lambda\phi$$

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From spectral theorem:

$$A\phi = \lambda\phi \implies A\phi - \lambda I\phi = 0$$

$$(A - \lambda I)\phi = 0$$

$$\begin{bmatrix} 30 - \lambda & 28 \\ 28 & 30 - \lambda \end{bmatrix} = 0 \implies \lambda = 58 \text{ and } \lambda = 2$$

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$$\begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = 58 \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} \implies \phi_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

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$$\begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 2 \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} \Longrightarrow \phi_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

EXAMPLE

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From spectral theorem: $A\phi = \lambda\phi$

$$\phi_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \lambda_1 = 58 \qquad \phi_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \lambda_2 = 2$$

$$\phi = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

COVARIANCE

$$Cov(X, X) = \frac{1}{n}\sum_{i=1}^{n} X^2$$

assuming X is mean centered

$$Cov(X, X) = \frac{1}{n} X X^T$$

SPECTRAL THEOREM

If $A \in \mathbb{R}^{m\times n}$ is symmetric matrix, then the , $m \times m$ matrix AA^T

and the $n \times n$ matrix A^TA are both symmetric

We can apply Spectral theorem to the matrices AA^T and A^TA

Question: How are the eigenvalues and the eigenvectors of these matrices related?

SPECTRAL THEOREM

Using Spectral theorem

$$(A^T A)\phi = \lambda\phi$$

Multiply both sides by X

$$A(A^T A)\phi = A\lambda\phi$$

$$AA^T(X\phi) = \lambda(A\phi)$$

The matrices AA^T and $A^T A$ share the same nonzero eigenvalues

SPECTRAL THEOREM

Using Spectral theorem

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$$AA^T(X\phi) = \lambda(A\phi)$$

Conclusion:

The matrices AA^T and $A^T A$ share the same nonzero eigenvalues

To get an eigenvector of AA^T from $A^T A$ multiply ϕ on the left by A

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Very powerful, particularly if number of observations, n, and the number of features, m, are drastically different in size.

For PCA: $Cov(A, A) = AA^T$

SINGULAR VALUE DECOMPOSITION

Theorem : $$A_{mn} = U_{mm} \Sigma_{mn} V_{nn}^T$$

A - Rectangular matrix, $m \times n$

Columns of U are orthonormal eigenvectors of AA^T

Columns of V are orthonormal eigenvectors of $A^T A$

Σ is a diagonal matrix containing the square roots of eigenvalues from U or V in descending order

SVD - EXAMPLE

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_{3\times 2} = U_{3\times 3}\Sigma_{3\times 2}V_{2\times 2}^T$$

Columns of U are orthonormal eigenvectors of AA^T

$$U = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

SVD - EXAMPLE

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_{3\times2} = U_{3\times3}\Sigma_{3\times2}V_{2\times2}^T$$

Columns of V are orthonormal eigenvectors of A^TA

$$V^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

SVD - EXAMPLE

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_{3\times2} = U_{3\times3}\Sigma_{3\times2}V_{2\times2}^T$$

Σ is a diagonal matrix containing the square roots of eigenvalues from U or V in descending order

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

SVD - EXAMPLE

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_{3\times 2} = U_{3\times 3}\Sigma_{3\times 2}V^T_{2\times 2}$$

$$A = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$