CMSC 657, Introduction to Quantum Information Processing
Fall 2019, University of Maryland

Due: September 26th, 2019
Date: September 7, 2019

Assignment 2

Please submit it electronically to ELMS. This assignment is 10% in your total points. For the simplicity of the grading, the total points for the assignment is 100.

Problem 1. *The Hadamard gate and qubit rotations*

1. *(5 points)* Suppose that $(n_x, n_y, n_z) \in \mathcal{R}^3$ is a unit vector and $\theta \in \mathcal{R}$. Show that

$$e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)} = \cos(\tfrac{\theta}{2})\, I - i \sin(\tfrac{\theta}{2})\, (n_x X + n_y Y + n_z Z).$$

2. *(7 points)* Find a unit vector $(n_x, n_y, n_z) \in \mathcal{R}^3$ and numbers $\phi, \theta \in \mathcal{R}$ so that

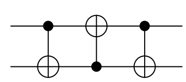
$$H = e^{i\phi} e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)},$$

where H denotes the Hadamard gate. What does this mean in terms of the Bloch sphere?

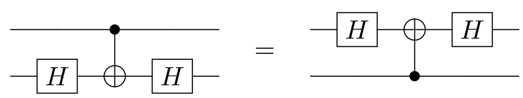
3. *(8 points)* Write the Hadamard gate as a product of rotations about the x and y axes. In particular, find $\alpha, \beta, \gamma, \phi \in \mathcal{R}$ such that $H = e^{i\phi} R_y(\gamma) R_x(\beta) R_y(\alpha)$.

Problem 2. *Circuit identities.*

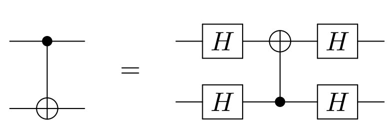
1. *(5 points)* Show that the following circuit swaps two qubits:



2. *(5 points)* Verify the following circuit identity:



3. *(5 points)* Verify the following circuit identity:



Give an interpretation of this identity.

Problem 3. *Universality of gate sets.* Prove that each of the following gate sets either is or is not universal. You may use the fact that the set $\{\textsc{cnot}, H, T\}$ is universal.

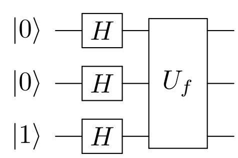
1. *(5 points)* $\{H, T\}$
2. *(5 points)* $\{\textsc{cnot}, T\}$
3. *(5 points)* $\{\textsc{cnot}, H\}$
4. *(5 points)* $\{\textsc{c}Z, K, T\}$, where $\textsc{c}Z$ denotes a controlled-Z gate and $K = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$
5. *(Bonus: 10 points)* $\{\textsc{cnot}, H, T^2\}$

Problem 4. *One-out-of-four search.* Let $f\colon \{0,1\}^2 \to \{0,1\}$ be a black-box function taking the value 1 on exactly one input. The goal of the one-out-of-four search problem is to find the unique $(x_1, x_2) \in \{0,1\}^2$ such that $f(x_1, x_2) = 1$.

1. *(2 points)* Write the truth tables of the four possible functions f.
2. *(3 points)* How many classical queries are needed to solve one-out-of-four search?
3. *(7 points)* Suppose f is given as a quantum black box U_f acting as

$$|x_1, x_2, y\rangle \overset{U_f}{\mapsto} |x_1, x_2, y \oplus f(x_1, x_2)\rangle.$$

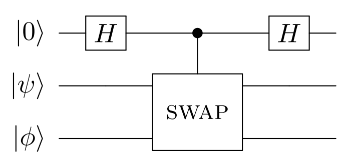
   Determine the output of the following quantum circuit for each of the possible black-box functions f:



4. *(3 points)* Show that the four possible outputs obtained in the previous part are pairwise orthogonal. What can you conclude about the quantum query complexity of one-out-of-four search?

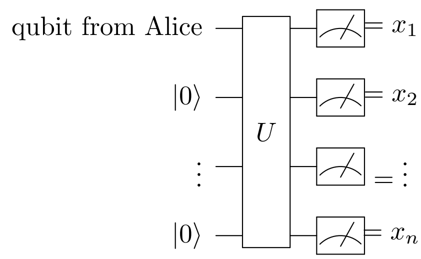
Problem 5. *Swap test.*

1. *(5 points)* Let $|\psi\rangle$ and $|\phi\rangle$ be arbitrary single-qubit states (not necessarily computational basis states), and let $\textsc{swap}$ denote the 2-qubit gate that swaps its input qubits (i.e., $\textsc{swap}|x\rangle|y\rangle = |y\rangle|x\rangle$ for any $x, y \in \{0,1\}$). Compute the output of the following quantum circuit:



2. *(5 points)* Suppose the top qubit in the above circuit is measured in the computational basis. What is the probability that the measurement result is 0?
3. *(3 points)* If the result of measuring the top qubit in the computational basis is 0, what is the (normalized) post-measurement state of the remaining two qubits?
4. *(2 points)* How do the results of the previous parts change if $|\psi\rangle$ and $|\phi\rangle$ are n-qubit states, and $\textsc{swap}$ denotes the $2n$-qubit gate that swaps the first n qubits with the last n qubits?

Problem 6. *A qubit cannot be used to communicate a trit perfectly* Suppose that Alice wants to convey a trit of information (an element of $\{0, 1, 2\}$) to Bob and all she is allowed to do is prepare one qubit and send it to Bob. Bob is allowed to prepare $n-1$ additional qubits, each in state $|0\rangle$, and apply an n-qubit unitary U operation to the entire n qubit system followed by a measurement in the computational basis.



The outcome will be an element of $\{0,1\}^n$. It is conceivable that such a scheme could exist where Bob can determine the trit from these n bits (e.g., by a function $f(x_1, \cdots, x_n) \in \{0,1,2\}$). We shall prove that this is impossible.

The framework is that Alice starts with a trit $j \in \{0, 1, 2\}$ (unknown to Bob) and, based on j, prepares a one-qubit state, $\alpha_j|0\rangle + \beta_j|1\rangle, j \in \{0, 1, 2\}$. and sends it to Bob.

Then Bob applies some n-qubit unitary U to $(\alpha_j|0\rangle + \beta_j|1\rangle)|00\cdots0\rangle$ and measures each qubit in the computational basis, obtaining some $x \in \{0,1\}^n$ as outcome. Finally, Bob applies some function $f : \{0,1\}^n \to \{0,1,2\}$ to x to obtain a trit. The scheme *works* if and only if, starting with any $j \in \{0, 1, 2\}$, the resulting x will satisfy $f(x) = j$ with probability 1.

1. *(5 points)* Note that each row of the matrix U is a 2^n-dimensional vector. For $j \in \{0, 1, 2\}$, define the space V_j to be the span of all rows of U that are indexed by an element of the set $f^{-1}(j) \subseteq \{0,1\}^n$. Prove that V_0, V_1, and V_2 are mutually orthogonal spaces.
2. *(5 points)* Explain why, for a scheme to work, $(\alpha_j|0\rangle+\beta_j|1\rangle)|00\cdots0\rangle \in V_j$ must hold for all $j \in \{0, 1, 2\}$.
3. *(5 points)* Prove that it is impossible for $(\alpha_j|0\rangle + \beta_j|1\rangle)|00\cdots0\rangle \in V_j$ to hold for all $j \in \{0, 1, 2\}$.