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Lecture notes for Quantum Approximate Optimization Algorithm (QAOA)

1 Background

As its name suggests, the quantum approximate optimization algorithm (QAOA) is a quantum algorithm for finding approximate solutions to optimization problems [1]. Common examples include constraint satisfaction problems, for example, MaxCut. QAOA can be thought of as a discretization of the quantum adiabatic algorithm (QAA or QADI), which uses adiabatic quantum computing to solve optimization problems.

Given a problem Hamiltonian, C, and a driving Hamiltonian, B, and a specified number of layers or steps, p, QAOA_p can be described by the following circuit,

$$|\gamma, \beta\rangle = e^{-i\beta_p B} e^{-i\gamma_p C} \cdots e^{-i\beta_1 B} e^{-i\gamma_1 C} |s\rangle \tag{1.1}$$

where $|s\rangle$ is the ground state of B, and where γ_i and β_i are user-defined parameters. In applying QAOA_p to an optimization problem, we wish to maximize (or minimize) $\langle\gamma, \beta| C |\gamma, \beta\rangle$, over all possible γ_i and β_i.

The problem Hamiltonian encodes the optimization problem such that the ground state of the Hamiltonian is the optimal solution to the optimization problem. Given a problem specified by n bits and m clauses, the objective is defined as,

$$C(z) = \sum_{\alpha=1}^{m} C_\alpha(z) \tag{1.2}$$

where z is an n bit string and $C_\alpha(z) = 1$ is z satisfies that clause, and is 0 otherwise.

For example in the case of MaxCut given a graph, $G = (V, E)$ ($n = |V|$, $m = |E|$), the following problem Hamiltonian is used,

$$C = \sum_{ij \in E} C_{ij} \quad \text{where } C_{ij} = \frac{1}{2}(I - Z_i Z_j) \tag{1.3}$$

A commonly used driving Hamiltonian is,

$$B = \sum_{j \in [n]} X_j \tag{1.4}$$

Thus, $|s\rangle = |+\rangle^{\otimes n}$, the uniform superposition of all computational basis state. Intuitively, this driving Hamiltonian "maps" all computational basis states to all other computational basis states equally.

For optimizing the parameters, γ_i and β_i, we consider the expectation of the Problem Hamiltonian in the output state of QAOA_p, which is given by,

$$F_p(\gamma, \beta) = \langle\gamma, \beta| C |\gamma, \beta\rangle \tag{1.5}$$

Let M_p be the maximum (or minimum) of F_p over all angles,

$$M_p = \max_{\gamma_i, \beta_i} F_p(\gamma, \beta) \tag{1.6}$$

Given γ and β vectors of $p-1$ angles for $p-1$ layers of QAOA. We can construct, γ' and β' vectors of p angles, by setting the last angles, γ_p and β_p, to be 0. This implies,

$$F_{p-1}(\gamma, \beta) = F_p(\gamma', \beta') \implies M_p \geq M_{p-1} \tag{1.7}$$

In other words with more layers, you have more degrees of freedom and thus, potentially better solutions.

This supposedly implies that,

$$\lim_{p \to \infty} M_p = \max_z C(z) \tag{1.8}$$

2 A Bounded Occurrence Constraint Problem

Max E3LIN2 is a combinatorial problem of bounded occurrences; given a set of linear equations containing exactly three boolean variables (E3) which sum to 0 or 1 mod 2 (LIN2), find a solution which maximizes the number of satisfied solutions. Each variable is guaranteed to be in no more than D equations.

One can use Gaussian elimination to determine if the set of linear equations has a solution; however, the problem becomes hard, when we wish to maximize the number of equations satisfied. Specifically, for general instances (without bounded occurrences) it has been shown that there is no efficient $(1/2+\epsilon)$ classical algorithm unless P=NP. Note, a random string is expected to satisfy half of the equations.

In the case we consider, where each variable is contained in most D clauses, there is a classical algorithm [3] for a similar but more general problem, Max-kXOR, with an approximation ratio of,

$$\frac{1}{2} + \frac{\text{constant}}{D^{1/2}} \tag{2.1}$$

It can be shown [2] that one layer of QAOA will efficiently produce a string that satisfies,

$$\frac{1}{2} + \frac{1}{101 D^{1/2} \ln D} \tag{2.2}$$

times the number of equations. However, in the typical case, the output string will satisfy

$$\frac{1}{2} + \frac{1}{2\sqrt{3e} D^{1/2}} \tag{2.3}$$

3 The Approximation Ratio for QAOA$_1$

In applying QAOA to this problem, we will use the standard driving hamiltonian,

$$B = \sum_{i=1}^{n} X_i \tag{3.1}$$

where n is the number of variables. Thus, the initial state will be the ground state of B, $|s\rangle = |+\rangle^{\otimes n}$. The objective operator for each linear equation of E3LIN2 can be written as,

$$\frac{1}{2}(1 \pm Z_a Z_b Z_c) \tag{3.2}$$

where a, b, c are the variables in the equation and the $\pm$ is determined by the sum of the equation.

Dropping the additive constant, which does not affect the solution to the problem, the problem Hamiltonian can be write as,

$$C = \frac{1}{2} \sum_{a<b<c} d_{abc} Z_a Z_b Z_c \tag{3.3}$$

where d_{abc} is zero if there is no equation containing a, b, c and ± 1 if there is and depending on the sum.

Using these Hamiltonians and some parameters, γ, β, the expected number of satisfied equations will be,

$$\frac{m}{2} + \langle -\gamma, \beta | \, C \, | -\gamma, \beta \rangle \tag{3.4}$$

where

$$| -\gamma, \beta \rangle = e^{-i\beta B} e^{i\gamma C} \, |s\rangle \tag{3.5}$$

For simplicity, we will set $\beta = \pi/4$. We will show that there exists $\gamma \in \left[\frac{-1}{10D^{1/2}}, \frac{1}{10D^{1/2}}\right]$, such that the expectation is,

$$\frac{1}{2} + \frac{1}{101 D^{1/2} \ln D} \tag{3.6}$$

Finding the actual value of γ for a specific problem can be done using an efficient search.

3.1 Proof of Bound

Here we'll will give the proof of the lower bound from [2]. The main idea in the proof is to break up the expectation into individual clauses (e.g. $Z_i Z_j Z_k$) and then see how the simplified observable interacts with the terms in the QAOA circuit. Specifically, only terms in B and C containing variables in the clause (e.g. $Z_i Z_j Z_k$) will contribute to the expectation. This will also us to simplify the general expectation into a trigonometric formula depending on fewer terms, which we will then be able to bound using some probability inequalities and facts about trigonometric functions.

Consider a specific clause, without loss of generality, we will say the clause contains variables 1, 2, and 3,

$$\frac{1}{2} d_{123} \, \langle s| \, e^{-i\gamma C} e^{i\beta B} Z_1 Z_2 Z_3 e^{-i\beta B} e^{i\gamma C} \, |s\rangle \tag{3.7}$$

All terms except for $X_1 + X_2 + X_3$ in B will compute with $Z_1 Z_2 Z_3$; furthermore, since $\beta = \pi/4$,

$$e^{i\beta(X_1+X_2+X_3)} Z_1 Z_2 Z_3 e^{-i\beta(X_1+X_2+X_3)} = \left(\prod_{i=1}^{3} \cos 2\beta + i \sin 2\beta X_1 \right) Z_1 Z_2 Z_3 = -Y_1 Y_2 Y_3 \tag{3.8}$$

Thus, the expectation (3.7) becomes,

$$\frac{1}{2} d_{123} \, \langle s| \, e^{-i\gamma C} e^{i\beta B} Y_1 Y_2 Y_3 e^{-i\beta B} e^{i\gamma C} \, |s\rangle \tag{3.9}$$

We can rewrite C, separating out the clause containing 1, 2, and 3,

$$C = \overline{C} + \frac{1}{2} d_{123} Z_1 Z_2 Z_3 \tag{3.10}$$

Now we can conjugate the $Y_1 Y_2 Y_3$ with the contribution from the clause 123, the expectation (3.7) becomes,

$$\frac{1}{2} d_{123} \, \langle s| \, e^{-i\gamma \overline{C}} \left(\cos(\gamma d_{123}) Y_1 Y_2 Y_3 + \sin(\gamma d_{123}) X_1 X_2 X_3 \right) e^{i\gamma \overline{C}} \, |s\rangle \tag{3.11}$$

We will first evaluate,

$$\langle s| \, e^{-i\gamma \overline{C}} X_1 X_2 X_3 e^{i\gamma \overline{C}} \, |s\rangle \tag{3.12}$$

By inserting $I = \sum_{x\in\{0,1\}} |x\rangle \langle x|$, we get

$$\sum_{z_1,z_2,z_3} \sum_{z_1',z_2',z_3'} \langle s| e^{-i\gamma\overline{C}} |z_1,z_2,z_3\rangle \langle z_1,z_2,z_3| X_1X_2X_3 |z_1',z_2',z_3'\rangle \langle z_1',z_2',z_3'| e^{i\gamma\overline{C}} |s\rangle \tag{3.13}$$

$$= \sum_{z_1,z_2,z_3} \langle s| e^{-i\gamma\overline{C}} |z_1,z_2,z_3\rangle \langle -z_1,-z_2,-z_3| e^{i\gamma\overline{C}} |s\rangle \tag{3.14}$$

Now, we will more closely consider $\overline{C}$. We will separate it into terms involving only Z_1, etc and Z_1Z_2, etc. Terms which involved none of 1, 2, and 3 will commute and cancel out. Thus, we can consider,

$$\overline{C} = Z_1C_1 + Z_2C_2 + Z_3C_3 + Z_1Z_2C_{12} + Z_1Z_3C_{13} + Z_2Z_3C_{23} \tag{3.15}$$

Note, both terms of Z_1Z_2, Z_1Z_3, and Z_2Z_3 will anticommute with $X_1X_2X_3$. Thus, they will also be cancelled out. Thus, only the first three terms of $\overline{C}$ contribute to the expectation; and the $X_1X_2X_3$ term of the expectation (3.12) becomes,

$$\frac{1}{8}\sum_{z_1,z_2,z_3} \langle \overline{s}| e^{-2\gamma(z_1C_1+z_2C_2+z_3C_3)} |\overline{s}\rangle \tag{3.16}$$

where $|\overline{s}\rangle = \prod_{a\in Q} |+\rangle_a$, where Q contains the terms appearing in C_1, C_2, and C_3. Let q be the size of Q, which can be as large as $6D$.

Writing out the above (3.16) and using some trigonometry identities, we get,

$$\frac{1}{4}\langle \overline{s}| [\cos(2\gamma(C_1+C_2+C_3)) + \cos(2\gamma(C_1-C_2-C_3)) \tag{3.17}$$

$$+ \cos(2\gamma(-C_1+C_2-C_3)) + \cos(2\gamma(-C_1-C_2+C_3))] |\overline{s}\rangle \tag{3.18}$$

Using,

$$C_i = \frac{1}{2}\sum_{a<b} d_{iab}Z_aZ_b \tag{3.19}$$

we can rewrite it (3.17) as,

$$\frac{1}{4\cdot 2^q}\sum_{z_a, a\in Q} [\cos(\gamma(c_1(z)+c_2(z)+c_3(z))) + \cos(\gamma(c_1(z)-c_2(z)-c_3(z))) \tag{3.20}$$

$$+ \cos(\gamma(-c_1(z)+c_2(z)-c_3(z))) + \cos(\gamma(-c_1(z)-c_2(z)+c_3(z)))] \tag{3.21}$$

We can think of each c_i as random variables from a distribution of q binary variables, then the contribution of the $X_1X_2X_3$ term (3.12) to the original expectation, is the following expectation,

$$\frac{1}{8}\mathbb{E}_z[\cos(\gamma(c_1+c_2+c_3)) + \cos(\gamma(c_1-c_2-c_3)) \tag{3.22}$$

$$+ \cos(\gamma(-c_1+c_2-c_3)) + \cos(\gamma(-c_1-c_2+c_3))] \tag{3.23}$$

The contribution from the $Y_1Y_2Y_3$ can be derived similarly and is,

$$\frac{1}{8}\mathbb{E}_z[\sin(\gamma(c_1+c_2+c_3)) + \sin(\gamma(c_1-c_2-c_3)) \tag{3.24}$$

$$+ \sin(\gamma(-c_1+c_2-c_3)) + \sin(\gamma(-c_1-c_2+c_3))] \tag{3.25}$$

Therefore, the expectation (3.7) becomes,

$$\frac{1}{8}d_{123}\mathbb{E}_z[\sin(\gamma(d_{123}+c_1+c_2+c_3)) + \sin(\gamma(d_{123}+c_1-c_2-c_3)) \tag{3.26}$$

$$+ \sin(\gamma(d_{123}-c_1+c_2-c_3)) + \sin(\gamma(d_{123}-c_1-c_2+c_3))] \tag{3.27}$$

Now, we consider the taylor expansion of the above in terms of γ, singling out the linear term,

$$\frac{1}{2}d_{123}^2\gamma + P_{123}^k(\gamma) + \mathcal{R}_{123}^k(\gamma) \tag{3.28}$$

where

$$P_{123}^k(\gamma) = \frac{1}{8}d_{123}\sum_{j=3,5,...}^{k}\frac{\gamma^j(-1)^{(j-1)/2}}{j!}\mathbb{E}_z[(d_{123}+c_1+c_2+c_3)^j + ...] \tag{3.29}$$

and

$$|\mathcal{R}_{123}^k| \leq \frac{1}{8}\frac{|\gamma|^{k+2}}{(k+2)!}\mathbb{E}_z[|d_{123}+c_1+c_2+c_3|^{k+1} + ...] \tag{3.30}$$

Note, that $d_{123}^2 = 1$ and P_{123}^k starts with a cubic term and has degree k, where k will depend on D. For any polynomial of degree 2, c, (e.g. c_i), we have the following fact,

$$\mathbb{E}_z[|c|^{k+2}] \leq (k+1)^{k+2}(\mathbb{E}_z[c^2])^{(k+2)/2} \tag{3.31}$$

Since $\mathbb{E}_z[c_i] = 0$,

$$\mathbb{E}_z[(d_{123}\pm c_1 \pm c_2 \pm c_3)^2] = 1 + \mathbb{E}_z[(c_1 \pm c_2 \pm c_3)^2] \tag{3.32}$$

Since $\mathbb{E}_z[c_i]^2 = \sum_{a<b} d_{iab}d_{iab} \leq D$,

$$\mathbb{E}_z[(d_{123}\pm c_1 \pm c_2 \pm c_3)^2] \leq 1 + 9D \tag{3.33}$$

Using the bound for polynomials of degree 2, this implies,

$$\mathbb{E}_z[|d_{123}\pm c_1 \pm c_2 \pm c_3|^{k+2}] \leq (k+1)^{k+2}(1+9D)^{(k+2)/2} \tag{3.34}$$

Therefore, using Stirling's formula and $e < 3$,

$$|\mathcal{R}_{123}^k| \leq \frac{1}{8}\frac{|\gamma|^{k+2}}{(k+2)!}(k+1)^{k+2}(1+9D)^{(k+2)/2} \tag{3.35}$$

$$\leq \frac{1}{2}\left(e(1+9D)^{1/2}|\gamma|\right)^{k+2} \tag{3.36}$$

$$\leq (9D^{1/2}|\gamma|)^{k+2} \tag{3.37}$$

Note, the original expecation can be written,

$$\langle -\gamma, \pi/4|\, C\, |-\gamma, \pi/4\rangle = \sum_{a<b<c}\frac{1}{2}\gamma + P_{abc}^k(\gamma) + \mathcal{R}_{abc}^k(\gamma) \tag{3.38}$$

$$= \frac{m}{2}\gamma + P^k(\gamma) + \sum_{a<b<c}\mathcal{R}_{abc}^k(\gamma) \tag{3.39}$$

where $P^k(\gamma)$ is the sum of the m polynomials.

By the triangle inequality,

$$|\langle -\gamma, \pi/4|\, C\, |-\gamma, \pi/4\rangle| \geq \left|\frac{m}{2}\gamma + P^k(\gamma)\right| - \left|\sum_{a<b<c}\mathcal{R}_{abc}^k(\gamma)\right| \tag{3.40}$$

$$\geq \left|\frac{m}{2}\gamma + P^k(\gamma)\right| - m(9D^{1/2}|\gamma|)^{k+2} \tag{3.41}$$

To keep the negative term small, let,

$$|\gamma| \leq \frac{1}{10D^{1/2}} \tag{3.42}$$

To lower bound the positive term, we use the following fact, for any constants a_i and odd k,

$$\max r = 0, 1, ..., k \left| x_r + a_2 x_r^2 + ... + a_k x_r^k \right| \geq \frac{1}{k} \tag{3.43}$$

where $x_r = cos(\pi r/k)$.

Thus, letting $\gamma_r = \frac{1}{10D^{1/2}} \cos(\pi r/k)$,

$$\max_{r=0,1,\ldots,k} \left| \frac{m}{2}\gamma_r + P^k(\gamma_r) \right| \geq \frac{m}{20D^{1/2}k} \tag{3.44}$$

Thus,

$$\max_{r=0,1,\ldots,k} \left| \frac{m}{2}\gamma_r + P^k(\gamma_r) \right| - m(9D^{1/2}|\gamma_r|)^{k+2} \geq \frac{m}{20D^{1/2}k} - m\left(\frac{9}{10}\right)^{k+2} \tag{3.45}$$

Letting $k = 5\ln D$, the right hand side is greater than,

$$\frac{m}{101D^{1/2}lnD} \tag{3.46}$$

Therefore,

$$\max_{r=0,1,\ldots,k} \langle -\gamma_r, \pi/4 | \, C \, | -\gamma_r, \pi/4 \rangle \geq \frac{m}{101D^{1/2}lnD} \tag{3.47}$$

where $\gamma_r \in \left[\frac{-1}{10D^{1/2}}, \frac{1}{10D^{1/2}}\right]$, which completes the proof of the bound.

3.2 The Average Case

A common practice in bounding the performance of an algorithm is to bound the expected approximation ratio in the average case; in other words, for a random problem. In this case for a fixed selection of equations, we will consider a random assignment for the sums of the equations, i.e. $d_{abc} = (x_a + x_b + x_c)$ mod 2. In other words, if there are m equations, there will be 2^m possible choices of all d_{abc}, and we will pick one uniformly at random.

Again considering the expectation for the term involving variables 1, 2, and 3, since the c_i do not involve d_{123}, the expected value of d_{123} is 0. Thus, the expectation can be written as,

$$\frac{1}{8}\sin\gamma \mathbb{E}_z[\cos(\gamma(c_1 + c_2 + c_3)) + \cos(\gamma(c_1 + c_2 + c_3)) \tag{3.48}$$

$$+ \cos(\gamma(c_1 + c_2 + c_3)) + \cos(\gamma(c_1 + c_2 + c_3))] \tag{3.49}$$

$$= \frac{1}{2}\sin\gamma \mathbb{E}_z[\cos(\gamma c_1)\cos(\gamma c_2)\cos(\gamma c_3)] \tag{3.50}$$

Now looking at the expectations of each of these terms with respect to d_{iab} for $i \in \{1, 2, 3\}$,

$$\mathbb{E}_d[\cos(\gamma c_i)] = \mathbb{E}_d\left[\gamma \sum_{3<a<b} d_{iab} z_a z_b\right] \tag{3.51}$$

$$= \frac{1}{2}\mathbb{E}_d\left[\prod_{3<a<b} \exp(i\gamma d_{iab} z_a z_b) + \prod_{3<a<b} \exp(-i\gamma d_{iab} z_a z_b)\right] \tag{3.52}$$

$$= \prod_{3<a<b} \cos(\gamma z_a z_b) \tag{3.53}$$

$$= \prod_{3<a<b} \cos\gamma \tag{3.54}$$

Note, the term does not depending on z. Since each c_i depend on distinct d's, the expected value over all the d's is,

$$\frac{1}{2}\sin\gamma(\cos\gamma)^{D_1+D_2+D_3} \tag{3.55}$$

where D_i is the number of terms in c_i, which implies $0 \leq D_1 + D_2 + D_3 \leq 3D$. Therefore, the contribution from every clause is,

$$\frac{m}{2}\sin\gamma\cos^{3D}\gamma \leq \mathbb{E}_d\left[\langle -\gamma, \pi/4 |\, C\, | -\gamma, \pi/4\rangle\right] \leq \frac{m}{2}\sin\gamma \tag{3.56}$$

Choosing γ to maximize the lower bound gives,

$$\gamma = \frac{g}{D^{1/2}} \tag{3.57}$$

For large D the lower bound is,

$$\frac{m}{2}\frac{g}{D^{1/2}}\exp\left(-\frac{3}{2}g^2\right) \tag{3.58}$$

where setting $g = 1/\sqrt{3}$ gives,

$$\frac{m}{2\sqrt{3e}D^{1/2}} \tag{3.59}$$

as desired.

It can be shown that for problem sizes of order m, the standard deviation of the expectation over all d's is of order $\sqrt{m}$. Thus, one can argue that with high probability, a "typical" instance will have at least the above approximation ratio minus a term of order $\sqrt{m}$.

4 A Classical Approximation Algorithm and Bound

In repsonse, to the first paper on QAOA's application to E3LIN2, i.e. Max-3XOR, which gave slightly worse bounds than those showed above, Barak et. all [3] gave a classical algorithm, which improved upon these bounds (and is still better than the above improved bounds for QAOA).

They showed that for any odd k, any instance of Max-kXOR, there is an efficient algorithm that finds an assignment satisfying,

$$\frac{1}{2} + \Omega\left(\frac{1}{\sqrt{D}}\right) \tag{4.1}$$

of the equations, where D is the max number of equations any one variable can be in. In the "triangle-free" case, they showed that there is an efficient algorithm which gives an assignment satisfying $\mu + \Omega(1/\sqrt{D})$, where μ is the number of equations that would be satisfied in a random assignment.

4.1 Proof of lower bound for $k = 3$

Here we will give the proof from [3] for the case of $k = 3$. The more general proof can be found in the paper. The paper uses a very common trick to lower bounding constraint satisfaction problems. First, they decouple the first coordinate. In other words instead of finding one solution (e.g x_i) for the n variables; one finds two solutions, y_i and z_i. The y_i are used for the first term of each equation and the z_i are used for the latter two terms. One can give a rounding scheme to convert, the y_i and z_i, into a single solution x_i, which only reduces the number of satisfied equations by a constant number. Finally, one can show that a random assignment of z_i (which then gives the y_i) gives the required lower bound.

To show the above, we will need some facts about the relationship between boolean functions and Fourier transformations. Any boolean function $f : \{\pm 1\}^n \to \mathbb{R}$ can be represented by a multilinear polynomial, i.e. the following Fourier expansion,

$$f(x) = \sum_{s\subset[n]} \hat{f}(S)x^S, \quad \text{where } x^S = \prod_{i\in S} x_i \tag{4.2}$$

Note, the following,

$$\mathbb{E}[f(x)] = \hat{f}(\varnothing) \tag{4.3}$$

$$\|f\|_2^2 = E[f(x)^2] = \sum_S \hat{f}(S)^2 \implies \mathrm{Var}[f(x)] = \sum_{S\neq\varnothing} \hat{f}(S)^2 \tag{4.4}$$

$$\inf_i[f] = \sum_{S\ni i} \hat{f}(S)^2 = \mathbb{E}[(\partial_i f)(x)^2] \tag{4.5}$$

Also, for any predicate $P : \{\pm 1\}^r \to 0, 1$, $r \geq 2$, we have $\mathrm{Var}[(\partial_i P)(x)] \geq \Omega(2^{-r})$ for all i.

Finally, we will use the fact that low-degree polynomials often achieve their expectation. Let $f : \{\pm\}^n \to \mathbb{R}$ be a multilinear polynomial of degree at most k. Then $\mathbb{P}[f(x) \geq \mathbb{E}[f]] \geq \frac{1}{4}$exp(-2k). In particular, for f^2,

$$\mathbb{P}[|f(x)| \geq \|f\|_2] \geq \exp(-O(k)) \tag{4.6}$$

which implies,

$$\mathbb{E}[|f(x)|] \geq exp(-O(k)) \cdot \|f\|_2 \geq exp(-O(k)) \cdot \mathrm{stddev}[f(x)] \tag{4.7}$$

In general, given an instance of the problem and an assignment $x \in \{\pm 1\}^n$, the number of satisfied constraints is,

$$\sum_{l=1}^{m} P_l(X_{S_l}) \tag{4.8}$$

which can be thought of as a multilinear function with degree at most k (in our case, 3).

To simplify things, we will instead consider the fraction of satisfied constraints (i.e. divide the above by m). Furthermore, we will replace P_l with $\overline{P}_l$,

$$\overline{P_l} = P_l - \mathbb{E}[P_l] = P_l - \hat{P}_l(\varnothing) \tag{4.9}$$

In this way, $\overline{P}_l$ can be thought of as the advantage over a random assignment.

Therefore, given an instance, we will defined the associated polynomial $\mathcal{B}(x)$ by,

$$\mathcal{B}(x) = \frac{1}{m}\sum_{l=1}^{m} \overline{P}_l(x_{S_l}) \tag{4.10}$$

Thus, in our case of $k = 3$, using the Fourier expansion,

$$\mathcal{B}(x) = \sum_{|S|=3} \hat{\mathcal{B}}(S)x^S = \sum_{i,j,k\in[n]} a_{ijk}x_i x_j x_k \tag{4.11}$$

where $\hat{\mathcal{B}}(S) \in \{\pm\frac{1}{2m}, 0\}$ depending on whether or not a constraint with those variables exists in the instance. Note, $a_{ijk} = \frac{1}{6}\hat{\mathcal{B}}(\{i, j, k\})$.

We will use the trick of "decoupling" the first coordinate, i.e. the algorithm will consider $\tilde{\mathcal{B}}(y, z) = \sum_{i,j,k} a_{ijk}y_i z_j z_k$ for some y_l and z_l. The algorithm will produce a good assignment for $\tilde{\mathcal{B}}$ and round to produce as assignment for the original using the following randomized rounding scheme,

$$\text{w.p. } \frac{4}{9}, x_i = \begin{cases} y_i \text{ w.p. } \frac{1}{2} \\ z_i \text{ w.p. } \frac{1}{2} \end{cases} \qquad \text{w.p. } \frac{4}{9}, x_i = \begin{cases} y_i \text{ w.p. } \frac{1}{2} \\ -z_i \text{ w.p. } \frac{1}{2} \end{cases} \qquad \text{w.p. } \frac{1}{9}, x_i = -y_i \tag{4.12}$$

Thus, the expectation is equal to,

$$\mathbb{E}[\mathcal{B}(x)] = \frac{4}{9}\sum_{i,j,k} a_{ijk}\left(\frac{y_i+z_i}{2}\right)\left(\frac{y_j+z_j}{2}\right)\left(\frac{y_j+z_j}{2}\right) \tag{4.13}$$

$$+\frac{4}{9}\sum_{i,j,k} a_{ijk}\left(\frac{y_i-z_i}{2}\right)\left(\frac{y_j-z_j}{2}\right)\left(\frac{y_j-z_j}{2}\right)+\frac{1}{9}\sum_{i,j,k} a_{ijk}(-y_i)(-y_j)(-y_k) \tag{4.14}$$

$$= \frac{1}{9}\sum_{i,j,k} a_{ijk}(y_iz_jz_k + z_iy_jz_k + z_iz_jy_k) \tag{4.15}$$

$$= \frac{1}{3}\tilde{\mathcal{B}}(y,z) \tag{4.16}$$

Now, we will write $\tilde{\mathcal{B}}(y,z) = \sum_i y_iG_i(z)$, where $G_i(z) = \sum_{j,k} a_{ijk}z_jz_k$. The algorithm just needs to find an assignment for z such that $\sum_i |G_i(z)|$ is large; then, we can take $y_i = \text{sgn}(G_i(z))$.

The algorithm simply chooses a random $z \in \{0,1\}^n$ uniformly. Thus, $\mathbb{E}[G_i(z)^2] = \sum_{j<k}(2a_{ijk})^2 = \frac{1}{9}\inf_i[\mathcal{B}]$. Applying the fact about low-degree polynomials achieving their expectations gives, $\mathbb{E}[|G_i(z)|] \geq \Omega(1)\cdot\sqrt{\inf_i[\mathcal{B}]}$. Finally, since $\inf_i[\mathcal{B}] = \deg(i)/4m^2$,

$$\mathbb{E}\left[\sum_i |G_i(x)|\right] \geq \Omega(1)\cdot\sum_i \frac{\sqrt{\deg(i)}}{m} \geq \Omega(1)\cdot\sum_i \frac{\deg(i)}{m\sqrt{D}} = \frac{\Omega(1)}{\sqrt{D}} \tag{4.17}$$

Since $\sum_i |G_i(x)|$ is bounded by 1/2, by Markov's inequality, the algorithm can find a z achieving $\sum_i |G_i(x)| \geq \Omega(1/\sqrt{D})$ with high probability after $O(\sqrt{D})$ many trials of z. Then, as stated above, y can be chosen to give $\tilde{\mathcal{B}}(y,z) \geq \Omega(1/\sqrt{D})$, which gives the correct lower bound for the expectation of $\mathcal{B}(x)$.

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