Quantum Games and Entanglement as a Resource

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Contents

1 Introduction

Bell's seminal paper [2] firmly established the non-local nature of quantum mechanics, disproving the existence of local hidden variable theories. This paved the way for understanding entanglement as a special and unique quantum phenomena, one without a classical equivalent. The idea of using quantum resources such as entanglement originates in works such as the CHSH paper [1] to test local hidden variable theories. The CHSH game further provided the idea of games as a unique platform on which to test and understand quantum foundations, since they highlight the non-classical aspects of quantum resources and entanglement, and emphasize the difference between quantum and classical randomness.

In this report, we identify formal ways of constructing and analyzing quantum games. We study early papers in the field of quantum games in order to understand how sharing quantum resources (i.e., entanglement) affects our game theoretic understanding - for example, how does sharing a quantum resource affect equilibrium in adversarial games. Specifically, we attempt to understand whether quantum resources can provide new Nash Equilibria [8] or even new forms of equilibria. Through our survey of these early papers, we build a foundation upon which to study new results, specifically aiming to quantify the effect of entanglement on equilibria.

We review and summarize multiple papers from the field of quantum game theory. First we discuss *Quantum Games and Quantum Strategies* [3] where the authors propose a way to model quantum games in general, and analyze the optimal strategies for a quantum version of the Prisoner's Dilemma. Even though one of their initial assumptions proved to be incorrect [9], the model they derived is useful for analyzing two-person binary-choice symmetric quantum games. Next, in *Quantum Strategies* [4], the authors discussed the PQ penny flip game and demonstrated that when one player uses quantum strategies, they can guarantee a win every time. They also developed three theorems that describe the effects of quantizing two-person zero-sum games.

We further study *Multiplayer Quantum Games* [5], which extends the formalism developed for 2 player games to games with multiple players. This paper shows that quantum resources can allow multiple players to develop non-classical strategies that, in effect, ensure co-operation and guarantee better outcomes. I.e, quantum entanglement provides a way for players to have trustworthy contracts.

After surveying these foundational papers, we seek to further understand the role of entanglement in changing the behaviour of games, motivated primarily by the fact that entanglement and the non-local correlations it provides allow quantum strategies to outperform classical strategies. Towards this goal, we discuss the paper *Connection Between Bell Nonlocality and Bayesian Game Theory* [10], which describes a relationship between certain types of games and Bell inequalities. We further study the behaviour of these strategies as well as equilibria at different levels of entanglement in [13]. Interestingly, some of these games show phase transitions in their behaviour (expected payoff and equilibrium) in terms of entanglement measure.

Quantum Games have also provided a useful way of understanding quantum devices and characterizing their behaviour. We lastly study [11] and [12], which describe experimental efforts towards implementing quantum games on a real quantum computer.

2 Motivation - CHSH game

The CHSH game is a way of formulating the CHSH inequality [1], which proposes an experiment to test Bell's theorem, i.e., the non-locality of quantum mechanics. This game is a 2 player co-operative Bayesian game - the players both aim to win together and they must execute their strategies under limited information. We denote the 2 players as Alice and Bob. Each player receives a bit from a referee - the bits for Alice and Bob are denoted x and y respectively. Alice and Bob must then produce a bit each, a, b respectively such that $a \oplus b = x \cdot y$. Neither player is allowed to communicate with the other, and x, y are chosen uniformly at random. The set up of the game is depicted in figure 1 below.

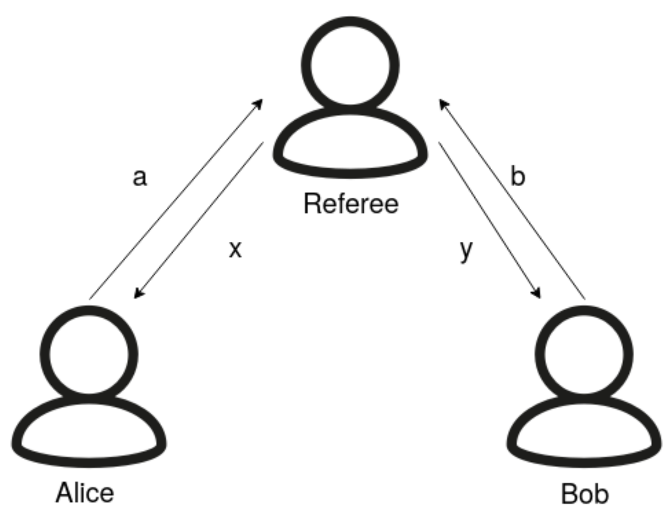

Figure 1: CHSH game

Classically, there are 4 possible inputs, and 4 possible outputs, leading to 16 possible strategies. Alice and Bob win in 3 of the possible input cases if their results agree (i.e, both 1 or both 0), and they win in the 4th case if their results disagree (We present only the possible outcomes, since $x \cdot y$ can only take on 0,1, while $a \oplus b$ can also only take on 0,1:

$x \cdot y$	(a, b)	Win?
0	(0, 1) or (1, 0)	No
0	(0, 0) or (1, 1)	Yes
1	(0, 1) or (1, 0)	Yes
1	(0, 0) or (1, 1)	No

$x \cdot y = 0$ only for 1 case out of the 4 possible inputs. We thus see that Alice and Bob can devise a strategy by which they win 75% of the time, merely by both putting out 0s (or 1s). By manually examining all the other 16 possible strategies, we see that this is optimal. Can this win probability be beaten by utilizing quantum mechanics?

In the quantum setting, Alice and Bob are allowed to share an entangled pair of qubits (a Bell pair). They are allowed to perform local unitaries and measurements on their own qubit, but not allowed to communicate classically. This allows Alice aand Bob to devise a strategy that allows them to outperform the best classical strategy:
Both Alice and Bob measure in the $\cos\theta\,|0\rangle + \sin\theta\,|1\rangle\,, \sin\theta\,|0\rangle - \cos\theta\,|1\rangle$ basis
Alice:
if $x = 0$, measure with $\theta_A = 0$, else, $\theta_A = \frac{\pi}{4}$
Bob:
if $y = 0$, measure with $\theta_B = -\frac{\pi}{4}$, else, $\theta_B = \frac{\pi}{8}$
The probability of Alice and Bob's measurements agreeing is given by $\cos^2\theta_A - \theta_B$. We see here that in the case where Alice and Bob must agree to win, the probability of agreeing is $\cos^2\frac{\pi}{8}$, and in the case where they must disagree to win, the probability of disagreeing is also $\cos^2\frac{\pi}{8}$. Thus the probability of Alice and Bob winning is always 88%, which is better than the best classical strategy! Quantum Entanglement thus provides Alice and Bob with a valuable resource that allows them to beat classical strategies. This motivates a further study of games which can have improved strategies given access to quantum resources.

3 Classical Game Theory

3.1 *Introduction to Classical Game Theory*

Before we get into quantum game theory, we will review some of the aspects of classical game theory. Game theory is a way to analyze various games. These games are constructed from rules, strategies, and outcomes which are all defined by clear mathematical parameters. There are various types of games. Some are cooperative where the players work together to "win" or achieve some result. There are also non-cooperative games where each player is working to try to maximize their individual payoff without concern for the other player's payoff. There are zero-sum games (a subset of non-cooperative) where one player's gain is another player's loss.

Some key ideas emerge when analyzing games. One is the concept of a Nash equilibrium [8]. A strategy is called a Nash equilibrium if, after picking that strategy, no player can increase their payoff by changing to a different strategy (assuming all other players' strategies are fixed). Another concept is called Pareto optimality [7]. This is when no individual can improve their payoff without another player losing something. The resources (payoff, points, etc.) are allocated optimally.

Next we will discuss a well-known game "The Prisoner's Dilemma" in the classical sense and then later introduce the quantum version.

3.2 *The Prisoner's Dilemma*

In the classical prisoner's dilemma, there are two players who are trying to maximize their individual payoffs. They each have two choices and are unaware of the other player's choice. They can choose to cooperate or defect. The outcomes are shown in the table below.

	Bob C	Bob D
Alice C	\$A=3, \$B=3	\$A=0, \$B=5
Alice D	\$A=5, \$B=0	\$A=1, \$B=1

The dilemma is that if they coordinated their efforts they could both walk away with a payoff of 3. However, since they are individually trying to maximize their payoff, they each must choose to defect which only gives them a payoff of 1. Another way to think about it is if Bob chooses to cooperate, then Alice should choose defect to get the maximum payoff. If Bob chooses to defect, then again Alice should choose defect to get the maximum payoff. The strategy "Defect-Defect" (DD) is considered a Nash equilibrium because either player would do worse if they decided to change their strategy.

4 Quantum Games

4.1 *Quantum Prisoner's Dilemma*

In [3], the authors discuss the well-known game "Prisoner's Dilemma". They show how using quantum strategies affects the outcomes. They find that when entanglement is used, the Nash equilibrium changes. According to this paper, in the quantum version of this game, DD is no longer a Nash equilibrium. However there is a Nash equilibrium that consists of quantum strategies.

The authors describe the quantum version of this game. In their formalism, each player has a 2-dimensional qubit which is located in the vector space spanned by orthonormal vectors $\{|C\rangle, |D\rangle\}$. $|C\rangle$ corresponds to cooperate and $|D\rangle$ corresponds to defect. Both qubits start in $|C\rangle$ and go through a 2-qubit unitary gate called $\hat{J}$. This gate can be tuned to provide no entanglement (identity gate) or maximum entanglement. Then the output goes through two one-qubit gates called $\hat{U}_A$ and $\hat{U}_B$. $\hat{U}_A$ is controlled by Alice based on the strategy she wants to use. $\hat{U}_B$ is controlled by Bob based on the strategy he wants to use. Classically, they can choose either

$$\text{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or D} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

However, those are obviously not the only unitary operations and in fact there are many more strategies available to them. Lastly, the qubits are sent through another gate $\hat{J}^\dagger$ and then they are measured to determine the payoff. The circuit for this setup is shown below.

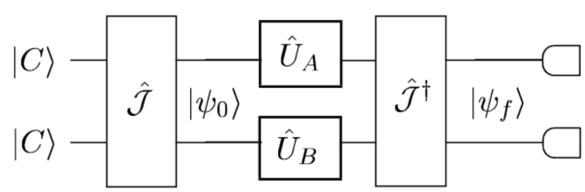

Figure 2: Quantum Prisoner's Dilemma Circuit

What they find is that when the $\hat{J}$ gate provides no entanglement, the game plays similarly to the classical version. The best strategy is still $D \otimes D$. However, when $\hat{J}$ entangles the qubits, if Bob plays $\hat{D}$ then Alice's best strategy is to play Q = $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Therefore, $D \otimes D$ is no longer a Nash equilibrium. However, $Q \otimes Q$ is a Nash equilibrium with a payoff of 3. So not only does the equilibrium change, but the best strategy allows for a higher payoff for each player. Using MATLAB, we were able to verify these findings as you can see in the graphs in figures 3 and 4.

After this paper was published, it was noted [9] that the authors did not account for the full range of unitary operations that Alice and Bob can use as their strategy. Because of this limitation, the authors incorrectly state that $Q \otimes Q$ is a Nash equilibrium. Actually, when you include all possible unitary operations

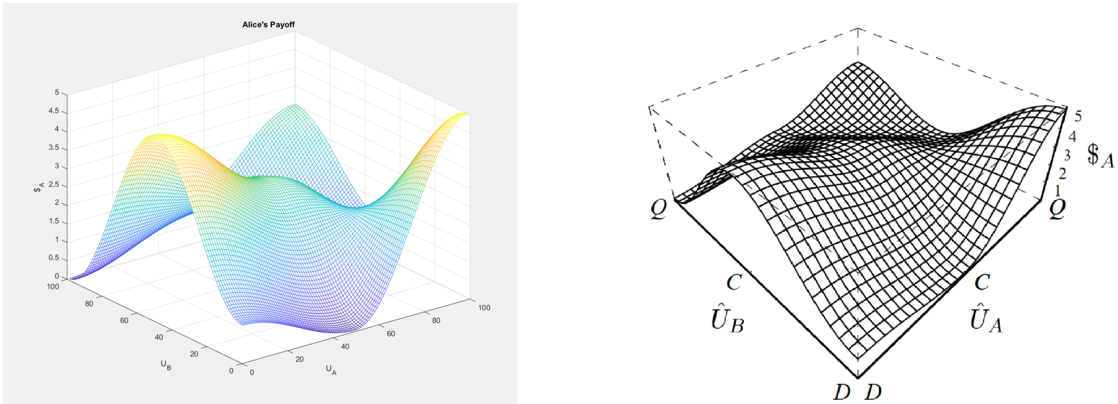

Figure 3: With entanglement - Our results (left) and results from paper (right)

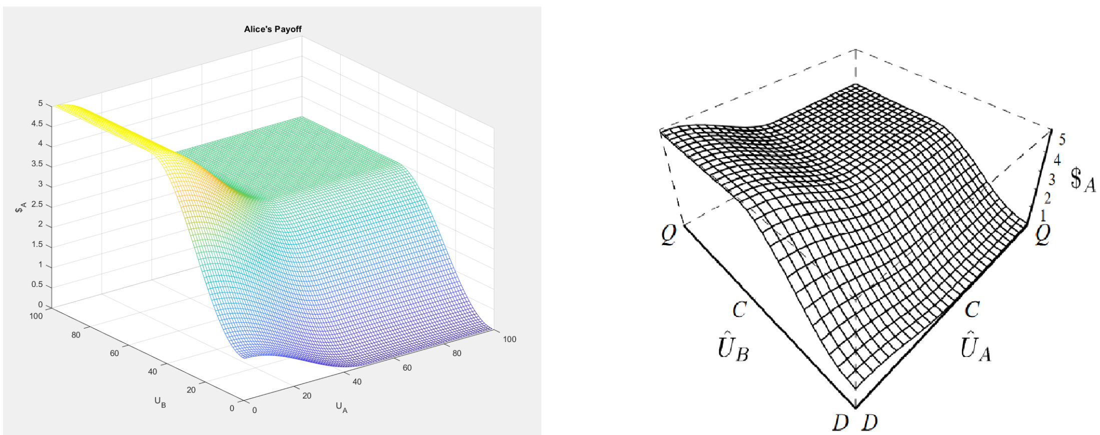

Figure 4: Without entanglement - Our results (left) and results from paper (right)

you find that there is no Nash equilibrium when the players are using pure quantum strategies. There is only an equilibrium when the players can make probabilistic choices to determine their quantum strategies.

4.2 *PQ Penny Flip*

The PQ penny flip game is a two-player zero-sum game. The idea is for the two players to start with a penny heads-up in a box. One player (Q) will choose whether to flip the penny or leave it be. The next player (P) will then make the same choice. Then Q will again choose whether to flip or not. The players do not know each other's decisions and Q wins if the penny is heads up after the three turns. If the penny is heads down then P wins. The reward table for P is shown below (1 indicates a win, -1 indicates a loss):

	Q=NN	Q=NF	Q=FN	Q=FF
P=N	-1	1	1	-1
P=F	1	-1	-1	1

In the classical version of this game, there is no deterministic Nash equilibrium. However, if each player plays a mixed strategy – one where they randomly (with probability $\frac{1}{2}$) decide whether to flip or not – then the expected payoff is zero. This is a probabilistic Nash equilibrium because their expected payoff would only decrease if either player changed the probability they use.

In [4], the authors extend this game into an unfair quantum version by restricting P to mixed classical strategies while letting Q use quantum strategies. They show that if this is the case, Q can win the game every time. The formalism is to represent the coin as a vector in the 2 dimensional vector space spanned by $\{|H\rangle, |T\rangle\}$, where $|H\rangle$ refers to heads-up and $|T\rangle$ refers to tails up. Then they show the two possible classical operations Flip (F) and No Flip (N):

$$\mathrm{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ or } \mathrm{N} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

These are the moves that P can do. The quantum strategy can be written as a unitary matrix

$$\mathrm{U(a,\ b)} = \begin{pmatrix} a & b^* \\ b & -a^* \end{pmatrix}$$

where $aa^* + bb^* = 1$. These are Q's available moves.

Since they are using a combination of quantum operations and classical operations, the state of the coin should be represented as a density operator ρ. The initial state of the coin would then be $|H\rangle \langle H|$. Q's optimal strategy is to use $\mathrm{U}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then applying this to the initial state of the coin gives $\rho = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then P applies the mixed operation $pF + (1-p)N$ to this state, where p is a probability chosen by P. However, it is easy to check that ρ is an eigenstate of both F and N, with eigenvalue 1. So ρ will be left unchanged regardless of P's action. Then Q only has to apply his unitary again to transform the state back into the pure state $|H\rangle \langle H|$. This way, Q wins every time. Since Q wins every time and P loses every time, both P and Q can do no better by unilaterally changing their strategy. Therefore this is an equilibrium.

The authors then prove three theorems regarding two-person zero-sum games:

1. A player using an optimal quantum strategy in a two-person zero-sum game has an expected payoff at least as great as his expected payoff with an optimal mixed strategy.

2. A two-person zero-sum game need not have a (quantum, quantum) equilibrium.

3. A two-person zero sum game always has a (mixed quantum, mixed quantum) equilibrium.

Last, an interesting side note is that the authors describe how this situation is analogous to quantum error corrections. If environmental noise can accidentally flip a qubit (the noise acts as P), then we can use the same quantum strategy to ensure the recovery of the initial state of the qubit.

4.3 *Multiplayer Quantum Games* [5]

Simon C. Benjamin and Patrick M. Hayden's paper Multiplayer quantum games was one of the first papers to explore the use of quantum information in games with more than 2 players. This paper showed the existence of new forms of equilibrium strategies in multiplayer games that do not appear in classical games or 2 player quantum games. As discussed, the comment to Quantum Games and Quantum Strategies showed that 2 player quantum games cannot have pure quantum equilibria (i.e., quantum 2 player games that do not allow mixed strategies do not admit equilibria). However, this paper showed that Multiplayer Quantum Games can exhibit equilibria for pure strategies, and that entanglement could be used as a guarantee for co-operative behaviour in games.

The authors of this paper adopt a similar formalism to those of Eisert et al [3]. Games are formally defined as involving agents/players, with a payoff function defined over player actions, and a strategy space of all available strategies. Equilibrium is defined in the same sense as inn classical game theory: strategies for each player that are stable, i.e., players do not stand to gain by changing their individual strategies. (such as, for example, the Nash Equilibrium).

The authors also adopt a similar quantization approach to Eisert et al [3]. Classical bits are generalized to qubits, which are entangled (but not necessarily maximally entangled), and the original classical game must be a special case of the resultant quantum game (when the qubits are not entangled, a quantum strategy leads to the same outcome as classical probabilistic strategies). The game proceeds as follows: players are given a qubit each (and may be allowed to use local ancillas, i.e., not shared across players). Players are allowed to locally manipulate their qubits, and the system is then disentangled, followed by a measurement which determines the players action. This framework allows quantum games to be seen as quantum algorithms.

Further considerations on the games affect whether or not both players are allowed quantum strategies. For example, in Meyer's Quantum Strategies [4], it is shown that quantum players can have an advantage over classical players (who are restricted to identity and bit flip gates, as opposed to universal single qubit unitaries). In this paper, both players are allowed to use universal single qubit unitaries, which is the most general form of quantum games. Quantum operations are represented as completely positive trace preserving maps, and strategies drawn from these maps are said to be in the space S_{TCP}. This paper shows that for the multiplayer Minority Game and a 3-player prisoner's dilemma, equilibria strategies can only be drawn from S_U (i.e, permitting only unitary maps, aka pure strategies), unlike in the 2 player cases, which forbid pure equilibria. (the 2 player cases, however, do permit mixed equilibria, which can be made to outperform or under-perform the classical case, depending on the payoff function).

We now describe the 2 games considered in this paper:

1. The N player Minority Game: In this game, players cast their votes (a binary choice), and the players who are in the minority are preferentially rewarded. In the event of an even split, there is no reward. When N=3, this paper shows that the game reduces to the classical version. However, for N=4, a quantum strategy allows players to achieve double the payoff in the quantum game than in the classical game, in a Nash equilibrium. This is an interesting result - as the number of players increases from 3 to 4, quantum resources allow the existence of a non-classical equilibrium that is better than the classical Nash equilibrium. However, this Nash equilibrium does not match the classical equilibrium with communication. This paper shows, however, that it is possible for a quantum strategy to outperform even a classical strategy with free communication, in the case of games with dominant strategies.

2. Dominant strategies are defined as strategies that yield higher payoffs than any other strategies, even accounting for the strategies of other players. For example, the dominant strategy in the classical 2 player Prisoner's Dilemma is (defect, defect). Here, the authors introduce a 3 player prisoner's dilemma, and show a Nash equilibrium strategy that outperforms the classical result even with communication (since communication does not change the dominant strategy in the absence of binding contracts). Beyond these 2 games, however, the authors do not guarantee existence of Nash equilibria in the quantum case, (and provide a counterexample where there any quantum equilibrium is worse than the classical equilibrium). In this case, entanglement hinders or 'spoils' the game.

5 Interesting Properties of Quantum Games

5.1 *Entanglement*

In [10], the authors show that there is a deep connection between Bell non-locality and Bayesian games. They first outline the definition of a Bayesian game, which is one where the players have incomplete information. Typically the players are assigned a type and based on that type, they will choose their strategy. The players do not know the type that the other players were assigned but they know the probability that the other players were assigned a specific type.

With this kind of game, the payoff function looks like this:

$$F_i = \sum \mu(X_1, X_2) P(A_1, A_2 | X_1, X_2) f_i(A_1, A_2, X_1, X_2)$$

where $\mu(X_1, X_2)$ is the probability that players 1 and 2 are assigned types X_1 and X_2, $P(A_1, A_2|X_1, X_2)$ is the probability that the the players will choose actions A_1 and A_2 given that they were assigned X_1 and X_2, and $f_i(A_1, A_2, X_1, X_2)$ is the payoff function for the given types and actions. The summation occurs over all the different types (X_1, X_2) and actions (A_1, A_2).

The key point is to notice how this summation is also a Bell expression. A Bell expression represents the correlations obtained in any experiment involving a classically correlated source. This can be represented as

$$S = \sum_{A_1,A_2,X_1,X_2} \alpha_{A_1,A_2,X_1,X_2} P(A_1, A_2|X_1, X_2)$$

where $P(A_1, A_2|X_1, X_2) = \int d\lambda p(\lambda)P(A_1|X_1, \lambda)P(A_2|X_2, \lambda)$.

In this expression, λ is a classically correlated variable. It has been proven [2] that this summation has an upper limit. This is called the Bell inequality. It has also been proven [1] that this inequality can be broken if the correlated variable is not classically correlated but is instead entangled (quantum correlation). Then, since the payoff function of a Bayesian game can be represented as a Bell expression (which has an upper bound), if we introduce quantum correlations then the players can get a higher payoff. An example of this is the CHSH game mentioned earlier in this paper.

One more point to make is that this paper discusses entanglement as a resource for games. In the Eisert model mentioned earlier, there is entanglement but it is in a different context - the entanglement is not used as a resource for the players to use to pick their strategy.

5.2 *Phase Transitions*

Most of these early papers study quantum games in the maximally entangled setting, i.e, when the players share a maximally entangled state. An interesting avenue of study is the behaviour of games and strategies when entanglement is treated as a resource (which we quantify through entanglement measure) and we examine the behaviour of games under varying levels of entanglement. We know that for games such as the Prisoner's dilemma, the optimal strategy in the maximally entangled regime can outperform the classical case. What happens to such games at thresholds, where the dominant strategy changes from classical to quantum (and vice versa)? This is explored in Du, J. et al [13], which examines the Prisoner's dilemma game with different payoff tables in both the classical and quantum setting. This study finds that while classical games remain unaffected, quantum games undergo phase transitions in strategy with respect to the entanglement measure. This paper makes use of Eisert's quantization model (though this model is not fully general as per [9] in the 2 player setting. Figure 5 shows the expected payoff for player Alice as a function of the entanglement measure.

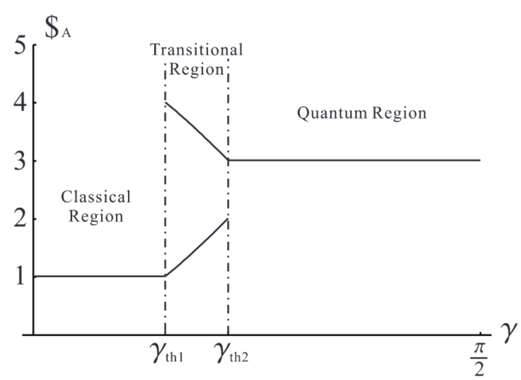

Figure 5: Phase transitions for Prisoner's Dilemma with 2 parameter unitaries

This result shows that equilibria and expected payoffs exhibit discontinuous dependence on entanglement measure, undergoing phase transitions for different levels of entanglement. Payoffs rapidly and instantaneously transition between best classical and best quantum results. In this particular case, we see a transition region where there are both classical and quantum equilibria. Interestingly, for some games (such as the prisoner's dilemma), there exist 'intermediate' or 'coexist' regions where the classical and quantum results show similar performance

Also of interest is the behaviour of optimal strategies in the fully general setting (with 3 parameter unitaries allowed). In this case, we know that in the quantum setting (maximally entangled), no pure Nash Equilibria are permitted. Figure 6 shows how the equilibrium abruptly disappears at the threshold of entanglement.

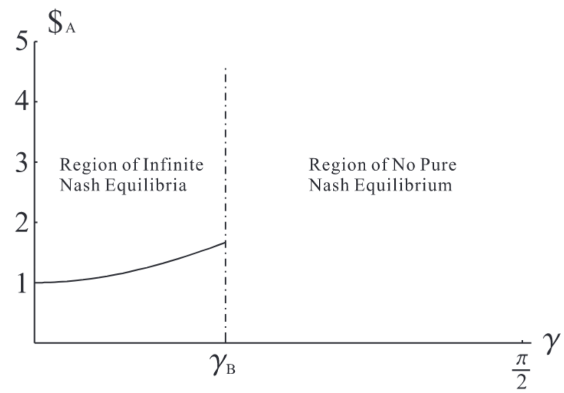

Figure 6: Phase transitions for Prisoner's Dilemma with 3 parameter unitaries

6 Implementations of Quantum Games

Although there has been plenty of research in designing and analyzing quantum models for different games, there has also been some research in implementing these models on real hardware. One interesting example is an ion-trap quantum computer that implements a modified version of the Quantum Prisoner's Dilemma [12]. In this experiment, they implement the following circuit which closely resembles the Eisert model:

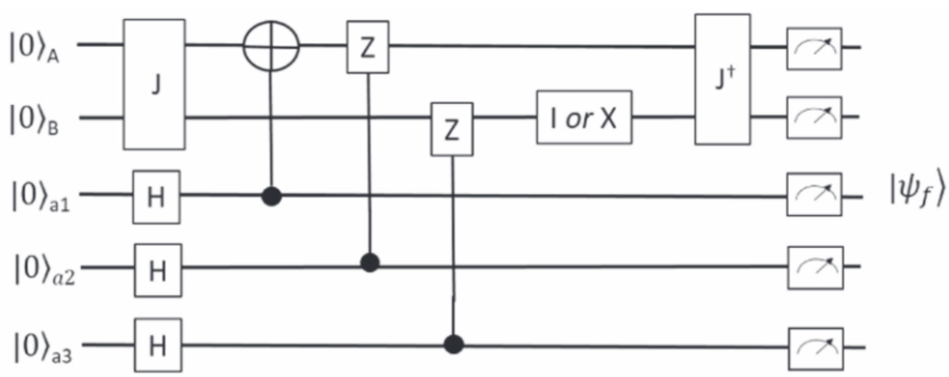

Figure 7: Circuit used to implement quantum game

The J and $J^\dagger$ gates are the entangling and disentangling gates. The top two qubits (A & B) represent the two players' strategies. The remaining gates are local unitaries to implement a strategy from the set $\{I, X, Y, Z\}$ and the bottom three qubits are ancillary qubits. The game being implemented is a modified version of the Prisoner's Dilemma. In this version, there are two payoff tables. The payoff table that is used is determined by some probability p. The two tables are:

	Bob (1) C	Bob (1) D
Alice C	\$A=11, \$B=9	\$A=1, \$B=10
Alice D	\$A=10, \$B=1	\$A=6, \$B=6

	Bob (2) C	Bob (2) D
Alice C	\$A=11, \$B=9	\$A=1, \$B=6
Alice D	\$A=10, \$B=1	\$A=6, \$B=0

The researchers ran this circuit many times with varying degrees of entanglement. They were able to theoretically calculate and experimentally determine the different Nash equilibria. They showed how the entangling gate affects the payoff at the Nash equilibrium as shown in figure 8 below. An interesting note is that the experiment found a Nash equilibrium that they did not predict theoretically (for χ around 2.5). Another interesting result from their experiment is that as the entanglement increases, their root-mean-

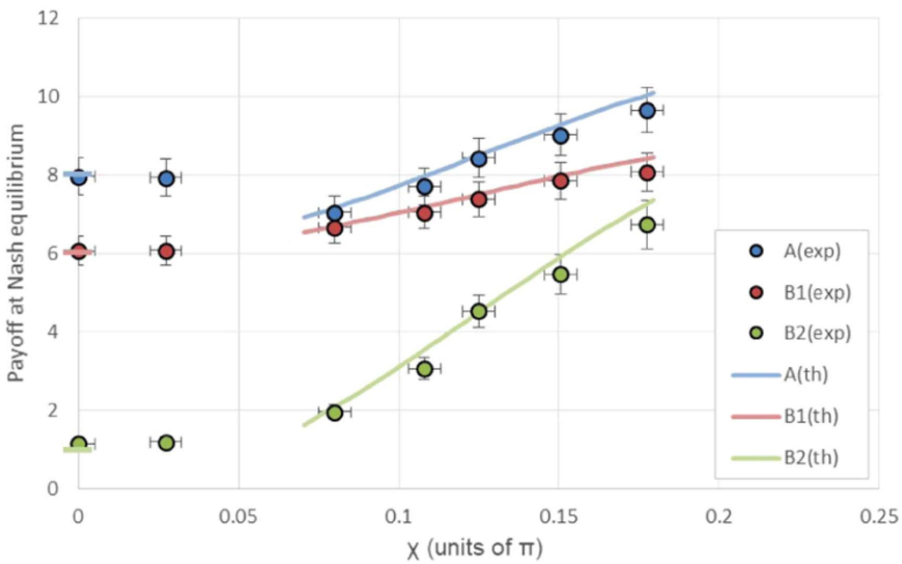

Figure 8: Payoff at Nash Equilibrium, χ is amount of entanglement

square deviation (a measure of the error) increases as well, as shown in figure 9 below. This shows the effect of using real gates which do not behave ideally.

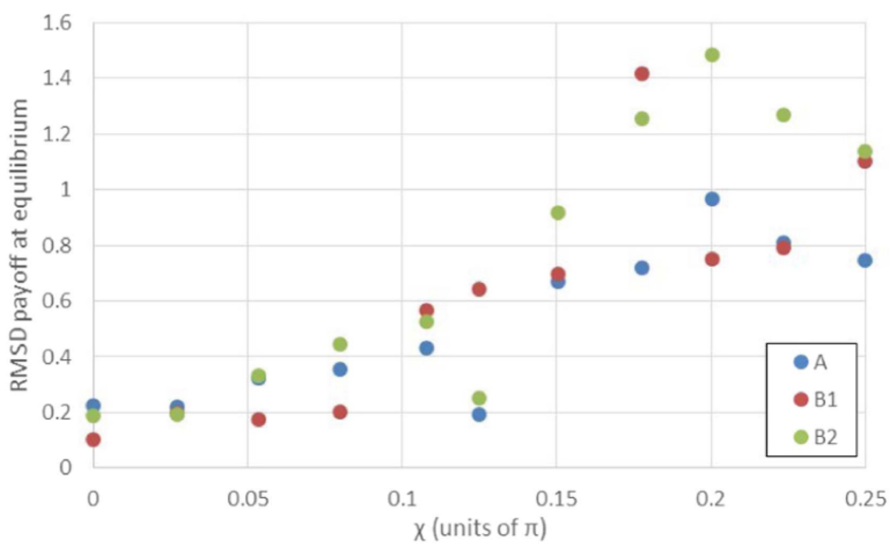

Figure 9: RMSD of payoff at Nash Equilibrium. $p = 0$

Finally, it should be mentioned that implementing games on a real quantum computer can help analyze the performance of the computer. You can compare gate fidelity by looking at a graph like in figure 9 which shows how the gate increases the error.

Another paper [11] also shows an implementation of the Eisert model with the Prisoner's Dilemma. They used a two-qubit NMR quantum computer to implement this circuit and they were able to experimentally verify the phase-transition behavior of the payoff as a function of entanglement, see figure 10 below.

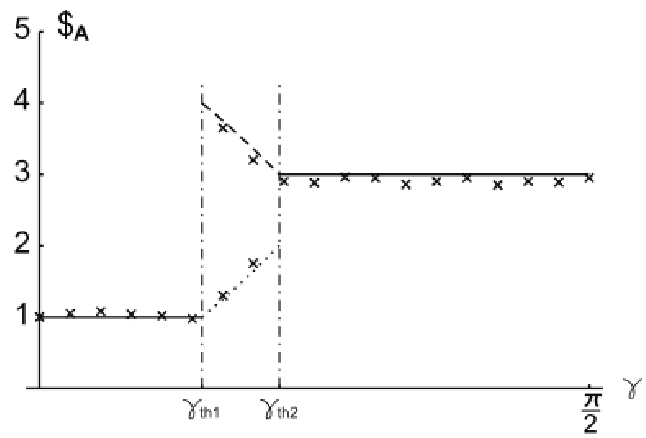

Figure 10: Expected payoff for Alice as a function of entanglement. Solid line is theory, x's are experiment

7 Open questions and future directions

These foundational papers have provided us an understanding of the formalism and the beginnings of quantum game theory, and have given us a game-theoretic framework to understand the use of entanglement as a resource in quantum games. As such, the field of quantum game theory has greatly developed in the last two decades, and there are further developments for us to explore in order to understand the state of the art of the field. In [6], we saw that when players only share partial or weak entanglement, there can exist a threshold for the entanglement measure below which it is optimal for players to select the classical dominant strategy, rather than the quantum strategy. (This follows from setting the entanglement measure to 0, which ensures that players must follow the classical game). An interesting direction to understand would be whether a threshold always exists (and for what types of games), and what sort of behaviour games have near this threshold (under what condition do optimal strategies undergo a sharp phase transition?). Another area to research is the effect of noise and decoherence on quantum games - i.e., given a non-ideal quantum device, how can we optimize the strategies for a maximal payoff? Last, we can explore and better understand the relationship between quantum games and complexity theory.

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