CMSC 330: Organization of Programming Languages

DFAs, and NFAs, and Regexps

The story so far, and what's next

- Goal: Develop an algorithm that determines whether a string *s* is matched by regex *R*
  - I.e., whether *s* is a member of *R*'s *language*

- Approach: Convert *R* to a finite automaton *FA* and see whether *s* is accepted by *FA*
  - Details: Convert *R* to a *nondeterministic FA* (NFA), which we then convert to a *deterministic FA* (DFA),
    - which enjoys a fast acceptance algorithm

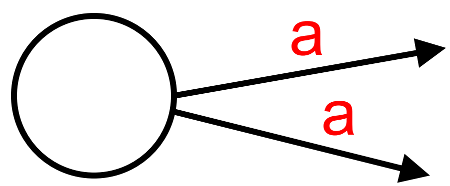
Two Types of Finite Automata

- Deterministic Finite Automata (DFA)
  - Exactly one sequence of steps for each string
    - Easy to implement acceptance check
  - All examples so far

- Nondeterministic Finite Automata (NFA)
  - May have many sequences of steps for each string
  - Accepts if any path ends in final state at end of string
  - More compact than DFA
    - But more expensive to test whether a string matches

Comparing DFAs and NFAs

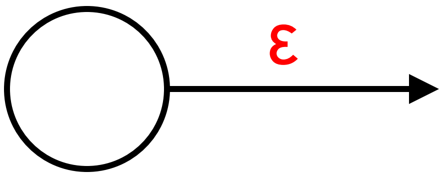
- NFAs can have more than one transition leaving a state on the same symbol



- DFAs allow only one transition per symbol
  - I.e., transition function must be a valid function
  - DFA is a special case of NFA

Comparing DFAs and NFAs (cont.)

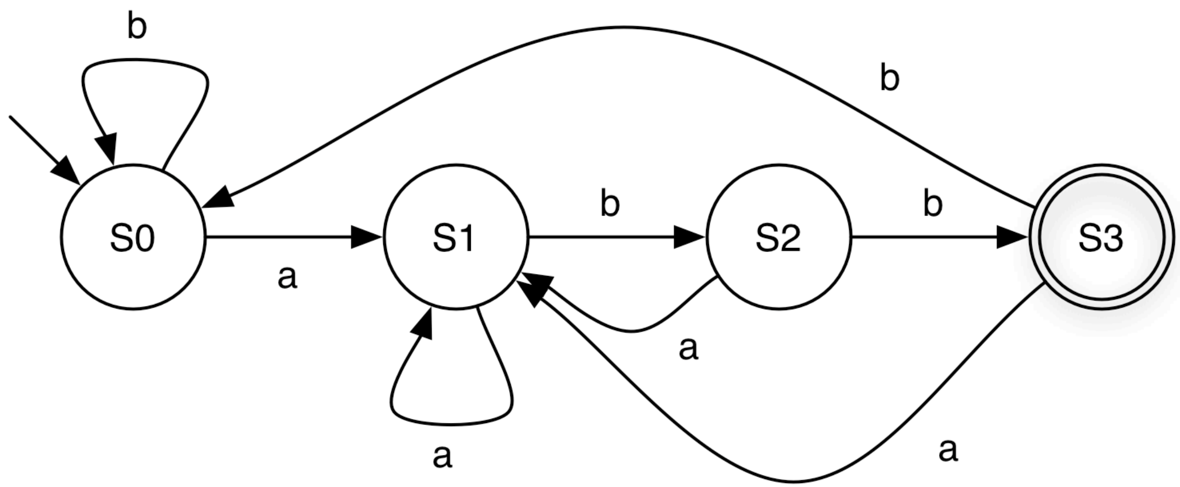
- NFAs may have transitions with empty string label
  - May move to new state without consuming character



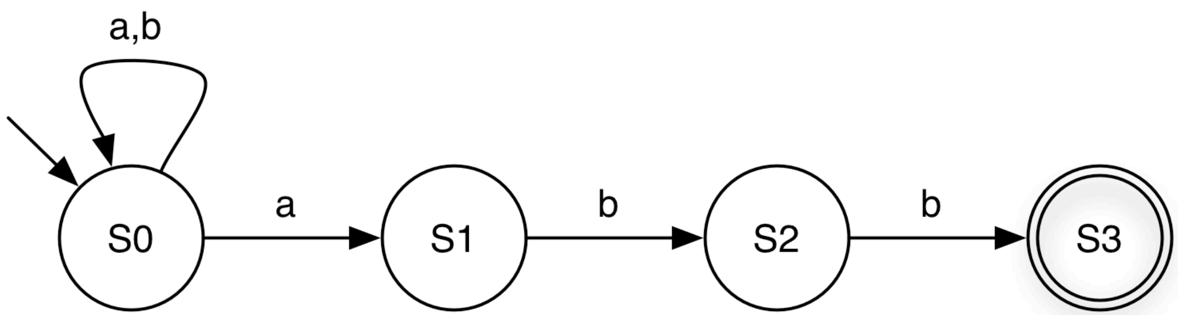
ε-transition

- DFA transition must be labeled with symbol
  - DFA is a special case of NFA

DFA for (a|b)*abb

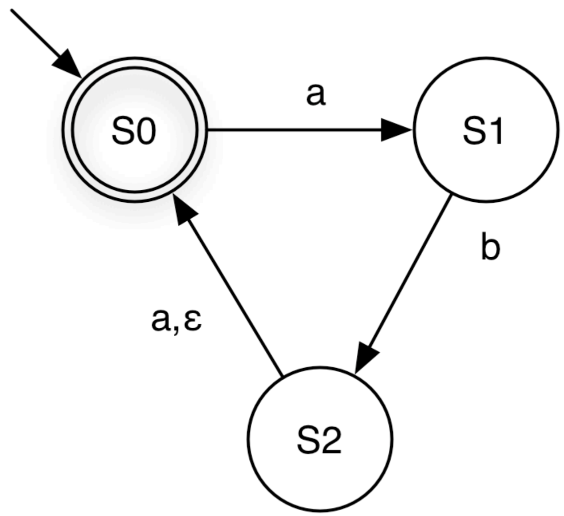


NFA for (a|b)*abb



- ba
  - Has paths to either S0 or S1
  - Neither is final, so rejected
- babaabb
  - Has paths to different states
  - One path leads to S3, so accepts string

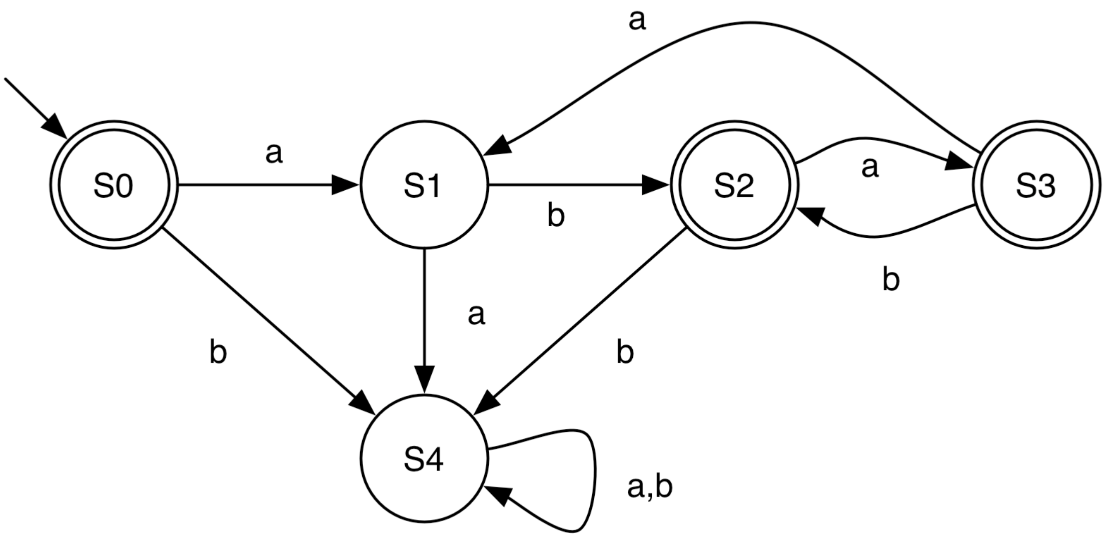
NFA for (ab|aba)*

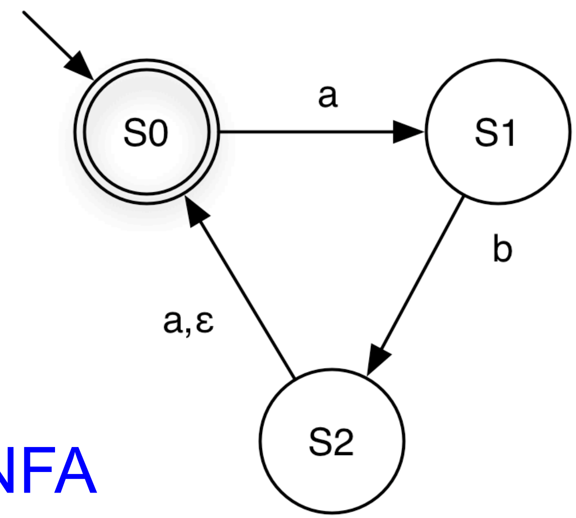


- aba
  - Has paths to states S0, S1
- ababa
  - Has paths to S0, S1
  - Need to use ε-transition

Comparing NFA and DFA for (ab|aba)*

DFA



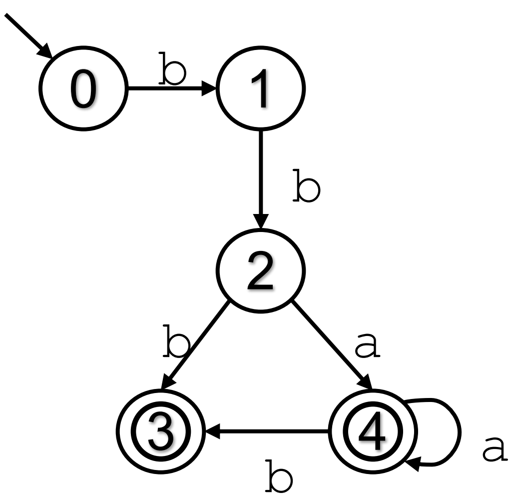


NFA

Quiz 1: Which DFA matches this regexp?

`b(b|a+b?)`

A.



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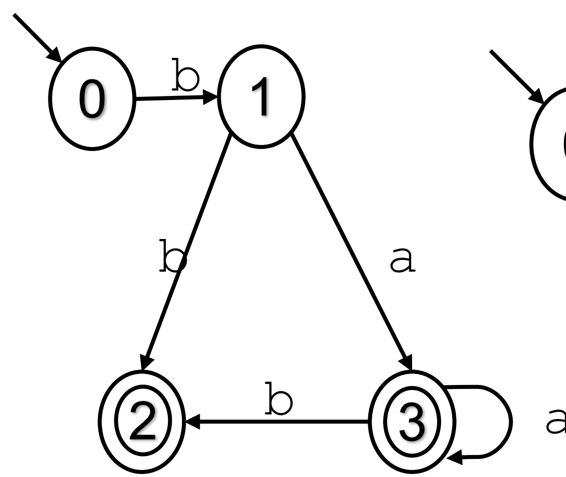

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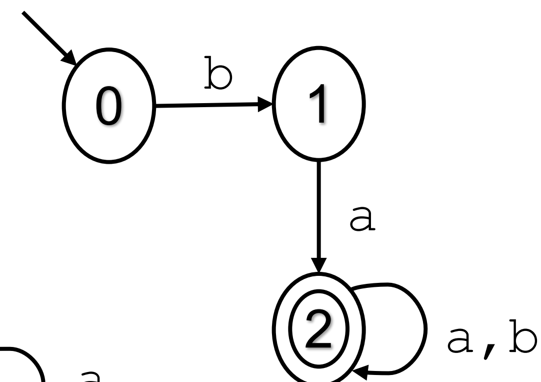

D. None of the above

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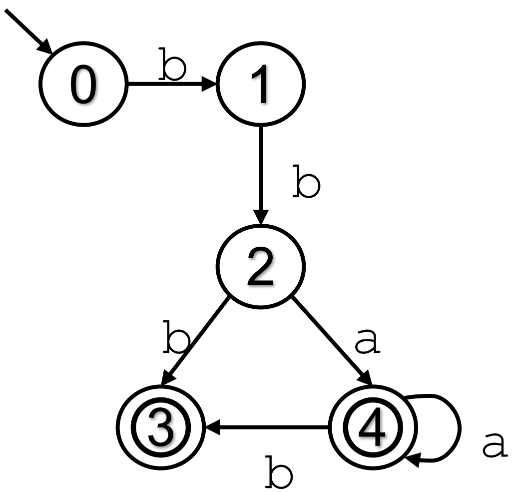

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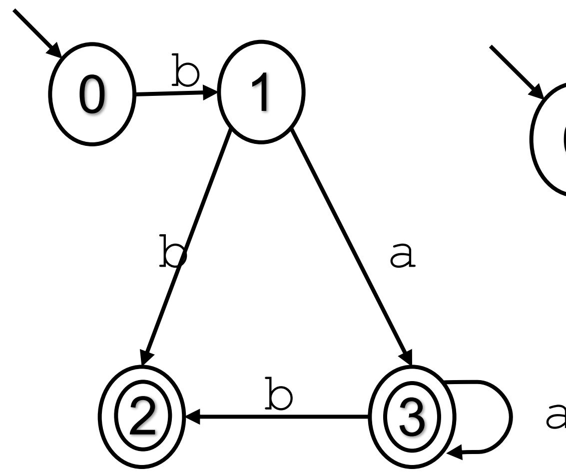
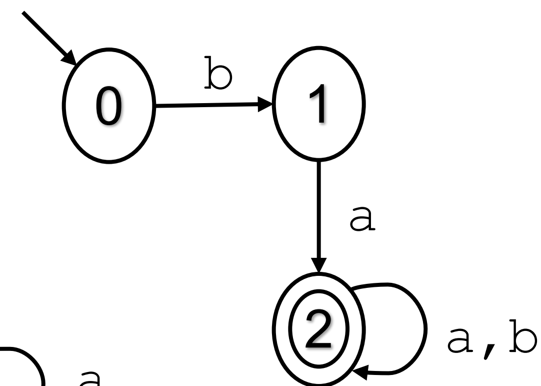


C.



D. None of the above

Formal Definition

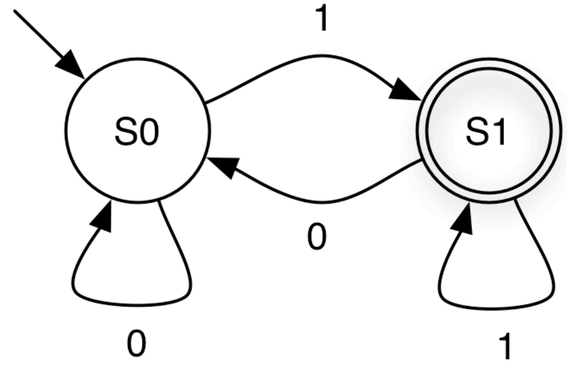
- A deterministic finite automaton *(DFA)* is a 5-tuple $(\Sigma, Q, q_0, F, \delta)$ where
  - Σ is an alphabet
  - Q is a nonempty set of states
  - $q_0 \in Q$ is the start state
  - $F \subseteq Q$ is the set of final states
  - $\delta : Q \times \Sigma \rightarrow Q$ specifies the DFA's transitions
    - What's this definition saying that δ is?
- A DFA accepts s if it stops at a final state on s

Formal Definition: Example

- $\Sigma = \{0, 1\}$
- $Q = \{S0, S1\}$
- $q_0 = S0$
- $F = \{S1\}$
-

symbol / input state

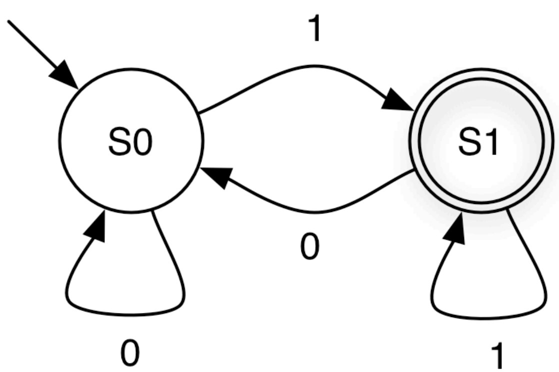
δ	0	1
S0	S0	S1
S1	S0	S1



or as { (S0,0,S0),(S0,1,S1),(S1,0,S0),(S1,1,S1) }

Implementing DFAs (one-off)

It's easy to build a program which mimics a DFA




```
cur_state = 0;
while (1) {

  symbol = getchar();

  switch (cur_state) {

    case 0: switch (symbol) {
              case '0':  cur_state = 0; break;
              case '1':  cur_state = 1; break;
              case '\n': printf("rejected\n"); return 0;
              default:   printf("rejected\n"); return 0;
            }
            break;

    case 1: switch (symbol) {
              case '0':  cur_state = 0; break;
              case '1':  cur_state = 1; break;
              case '\n': printf("accepted\n"); return 1;
              default:   printf("rejected\n"); return 0;
            }
            break;

    default: printf("unknown state; I'm confused\n");
             break;
  }
}
```

Implementing DFAs (generic)

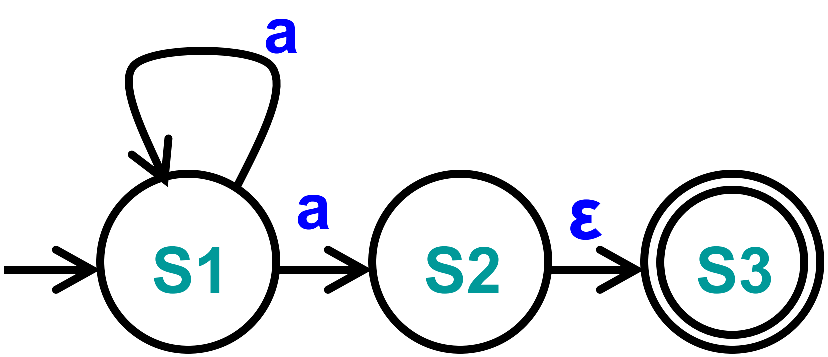
More generally, use generic table-driven DFA

```
given components (Σ, Q, q0, F, δ) of a DFA:
let q = q0
while (there exists another symbol σ of the input string)
    q := δ(q, σ);
if q ∈ F then
    accept
else reject
```

- q is just an integer
- Represent δ using arrays or hash tables
- Represent F as a set

Nondeterministic Finite Automata (NFA)

- An *NFA* is a 5-tuple $(\Sigma, Q, q_0, F, \delta)$ where
  - Σ, Q, q0, F as with DFAs
  - $\delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times Q$ specifies the NFA's transitions



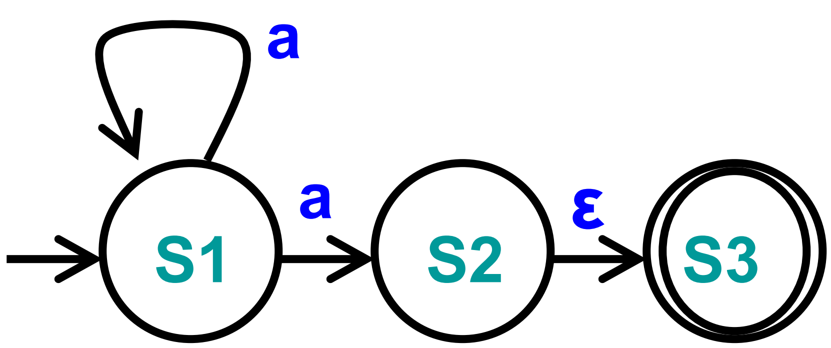
Example

- $\Sigma = \{a\}$
- Q = {S1, S2, S3}
- q_0 = S1
- F = {S3}
- δ = { (S1,a,S1), (S1,a,S2), (S2,ε,S3) }

- An NFA accepts s if there is at least one path via s from the NFA's start state to a final state

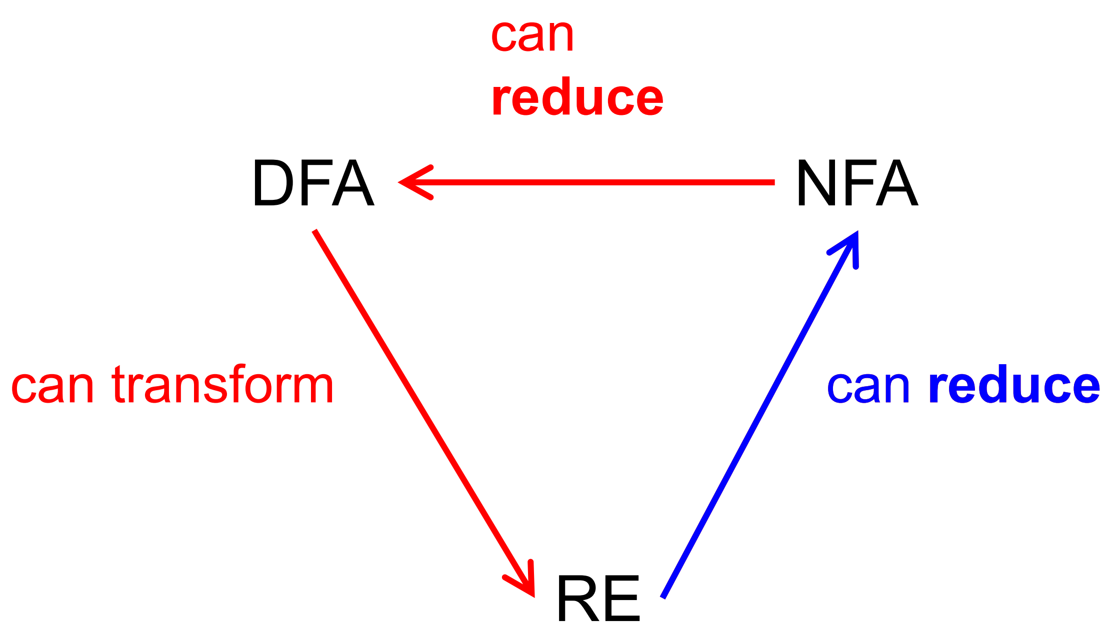
NFA Acceptance Algorithm (Sketch)

- When NFA processes a string s
  - NFA must keep track of several "current states"
    - Due to multiple transitions with same label, and ε-transitions
  - If any current state is final when done then accept s
- Example
  - After processing "a"
    - NFA may be in states
      S1
      S2
      S3
    - Since S3 is final, s is accepted
- Algorithm is slow, space-inefficient; prefer DFAs!



Relating REs to DFAs and NFAs

- Regular expressions, NFAs, and DFAs accept the same languages! *Can convert between them*



NB. Both *transform* and *reduce* are historical terms; they mean "convert"

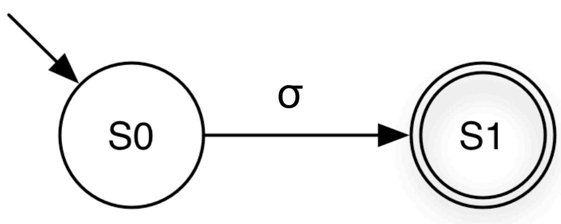
Reducing Regular Expressions to NFAs

- Goal: Given regular expression *A*, construct NFA: <*A*> = $(\Sigma, Q, q_0, F, \delta)$
  - Remember regular expressions are defined recursively from primitive RE languages
  - Invariant: $|F| = 1$ in our NFAs
    - Recall F = set of final states

- Will define <*A*> for base cases: σ , ε , ∅
  - Where σ is a symbol in Σ
- And for inductive cases: *AB*, *A*|*B*, *A**

Reducing Regular Expressions to NFAs

- Base case: σ

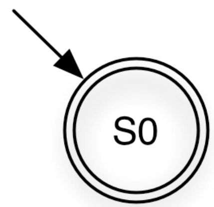


Recall: NFA is (Σ, Q, q_0, F, δ)
where
Σ is the alphabet
Q is set of states
q_0 is starting state
F is set of final states
δ is transition relation

<σ> = ({σ}, {S0, S1}, S0, {S1}, {(S0, σ, S1)})

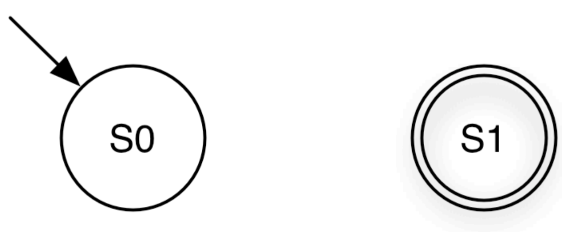
Reduction

- Base case: ε



$\langle \varepsilon \rangle = (\emptyset, \{S0\}, S0, \{S0\}, \emptyset)$

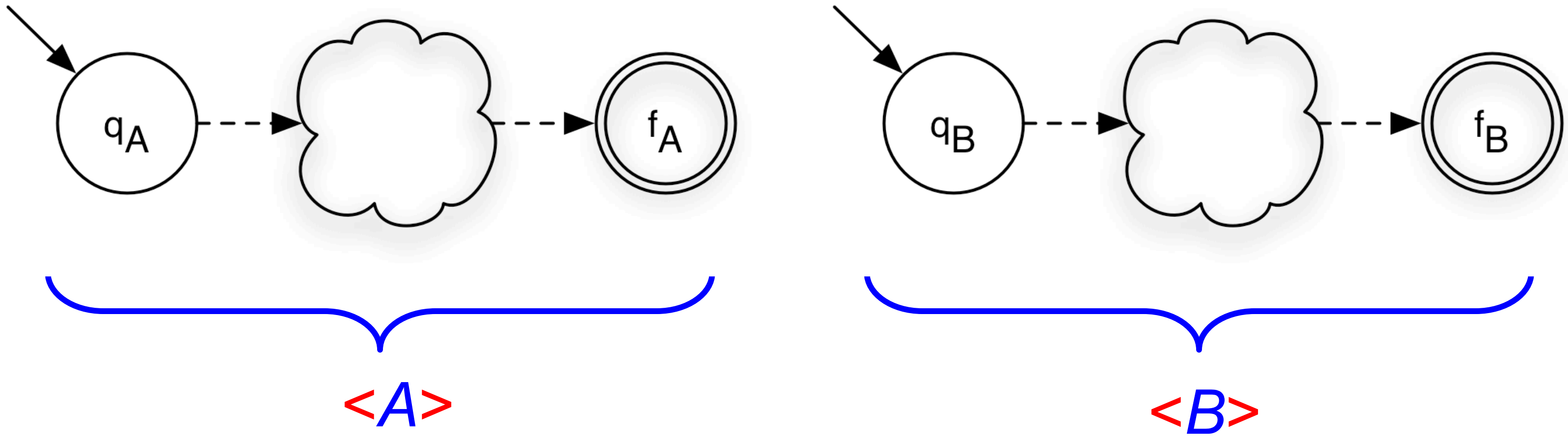
- Base case: ∅



$\langle \emptyset \rangle = (\emptyset, \{S0, S1\}, S0, \{S1\}, \emptyset)$

Reduction: Concatenation

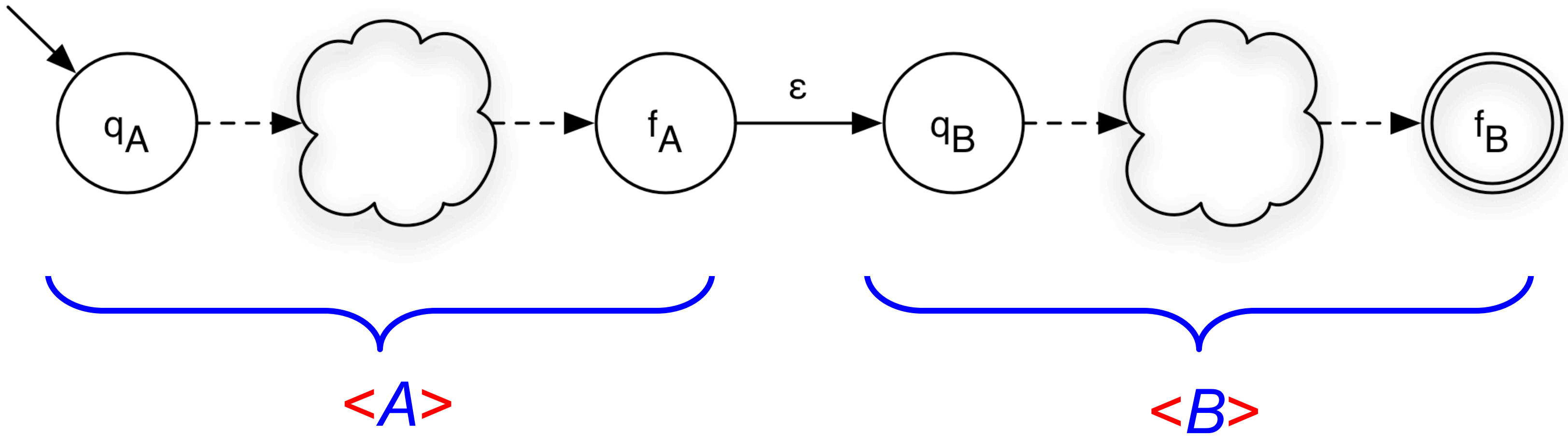
- Induction: *AB*



- $<A> = (\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
- $<B> = (\Sigma_B, Q_B, q_B, \{f_B\}, \delta_B)$

Reduction: Concatenation

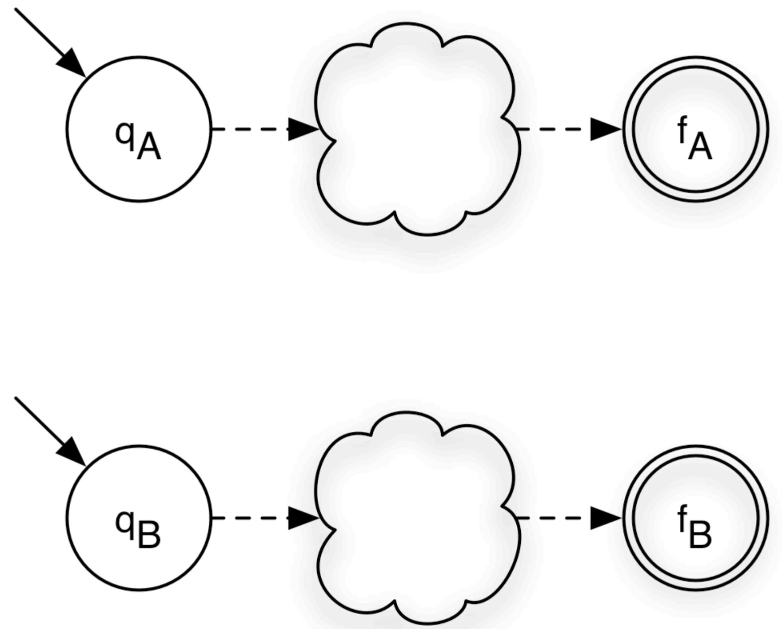
- Induction: *AB*



- <*A*> = $(\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
- <*B*> = $(\Sigma_B, Q_B, q_B, \{f_B\}, \delta_B)$
- <*AB*> = $(\Sigma_A \cup \Sigma_B, Q_A \cup Q_B, q_A, \{f_B\}, \delta_A \cup \delta_B \cup \{(f_A,\varepsilon,q_B)\})$

Reduction: Union

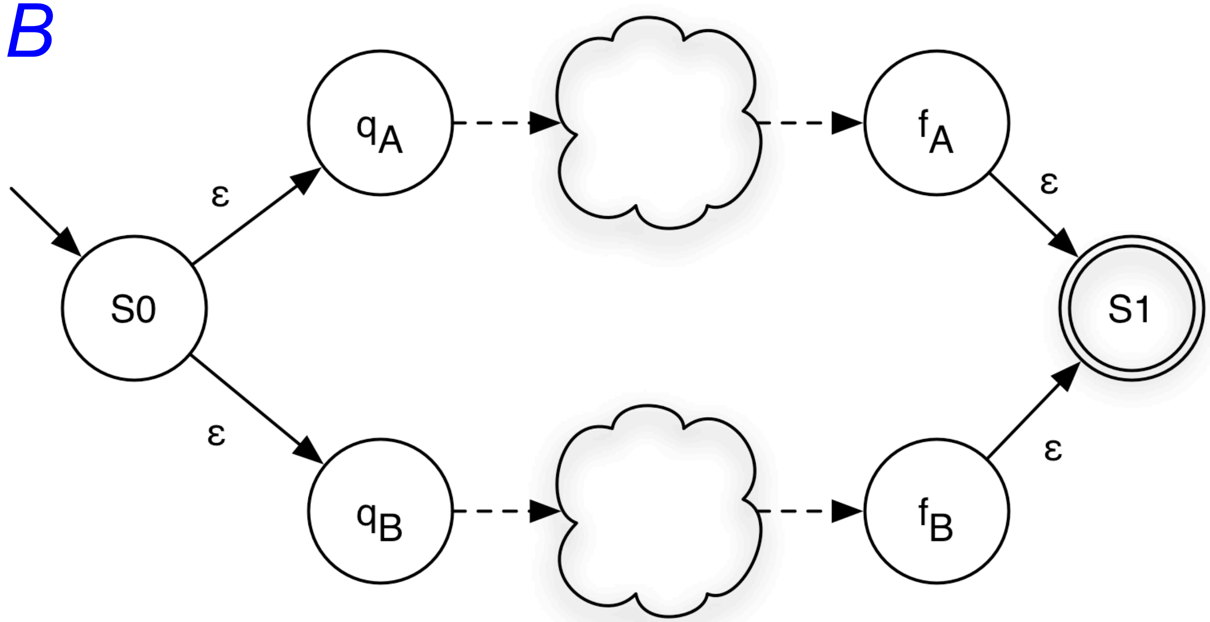
- Induction: *A*|*B*



  - <*A*> = $(\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
  - <*B*> = $(\Sigma_B, Q_B, q_B, \{f_B\}, \delta_B)$

Reduction: Union

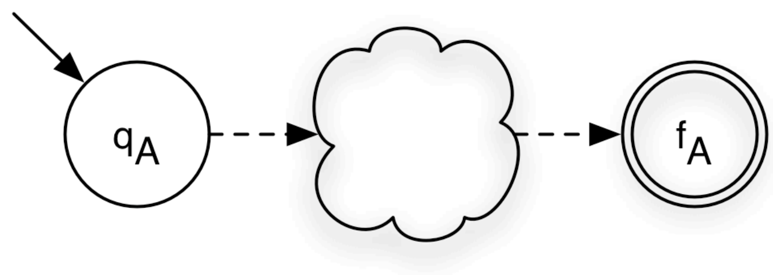
- Induction: *A|B*



- $<A> = (\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
- $<B> = (\Sigma_B, Q_B, q_B, \{f_B\}, \delta_B)$
- $<A|B> = (\Sigma_A \cup \Sigma_B, Q_A \cup Q_B \cup \{S0,S1\}, S0, \{S1\},$
  $\delta_A \cup \delta_B \cup \{(S0,\varepsilon,q_A), (S0,\varepsilon,q_B), (f_A,\varepsilon,S1), (f_B,\varepsilon,S1)\})$

Reduction: Closure

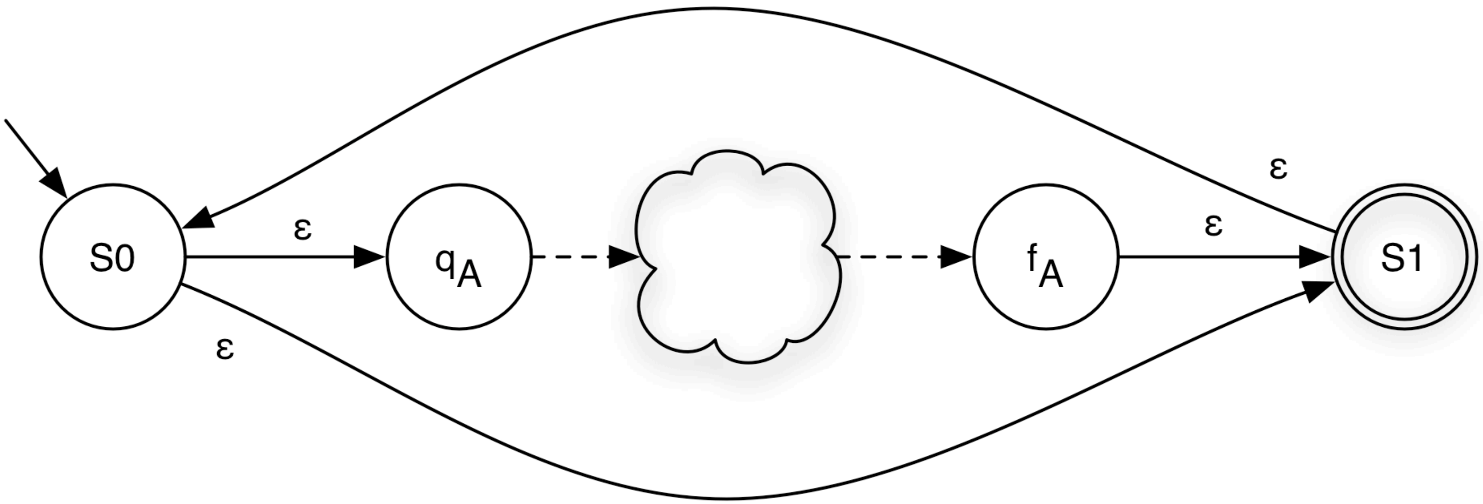
- Induction: *A**



- <*A*> = $(\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$

Reduction: Closure

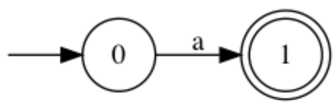
- Induction: *A**



- <*A*> = $(\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
- <*A**> = $(\Sigma_A, Q_A \cup \{S0,S1\}, S0, \{S1\},$
  $\delta_A \cup \{(f_A,\varepsilon,S1), (S0,\varepsilon,q_A), (S0,\varepsilon,S1), (S1,\varepsilon,S0)\})$

Quiz 2: Which NFA matches **a*** ?

A.



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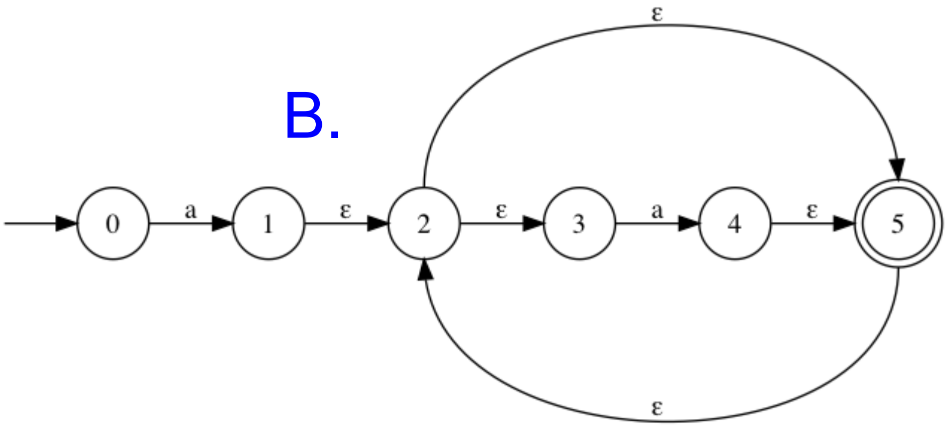

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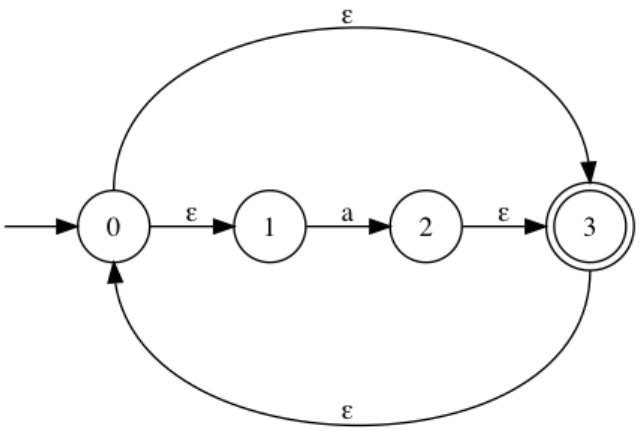

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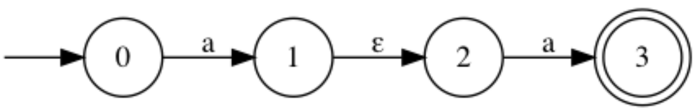

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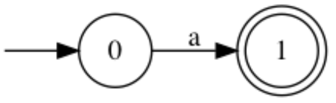

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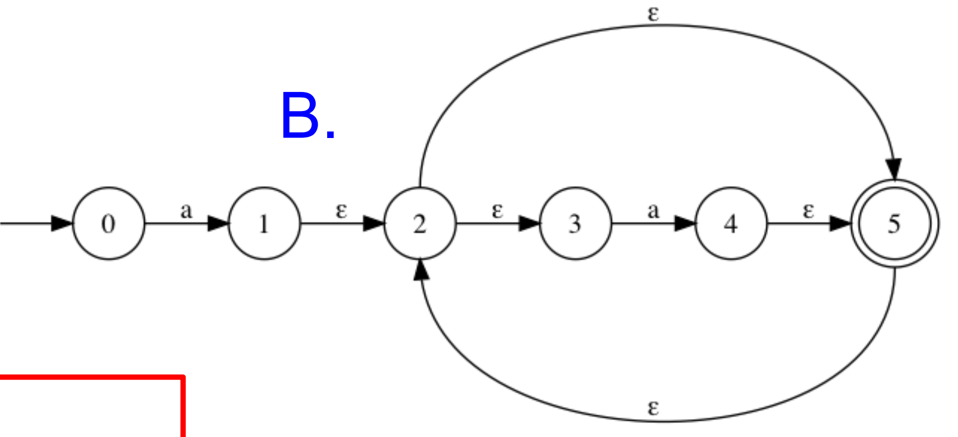

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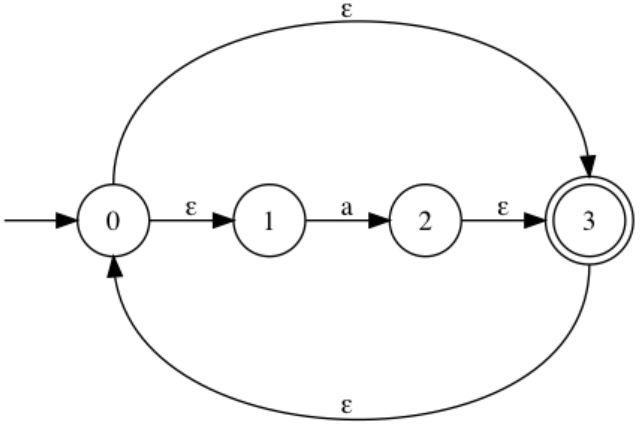

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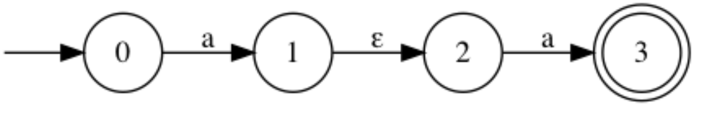

Quiz 3: Which NFA matches **a|b*** ?

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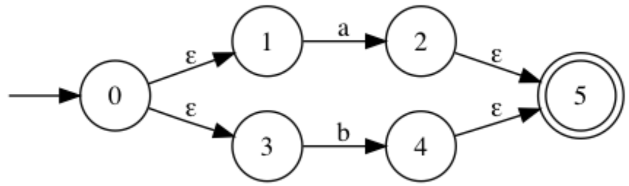

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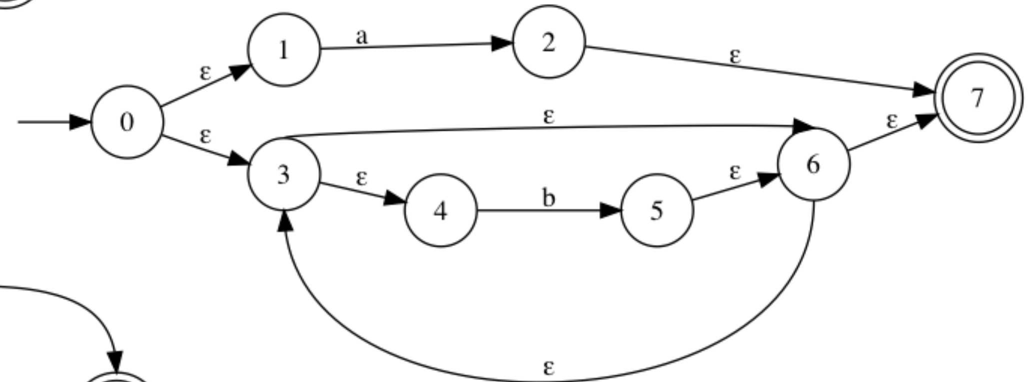

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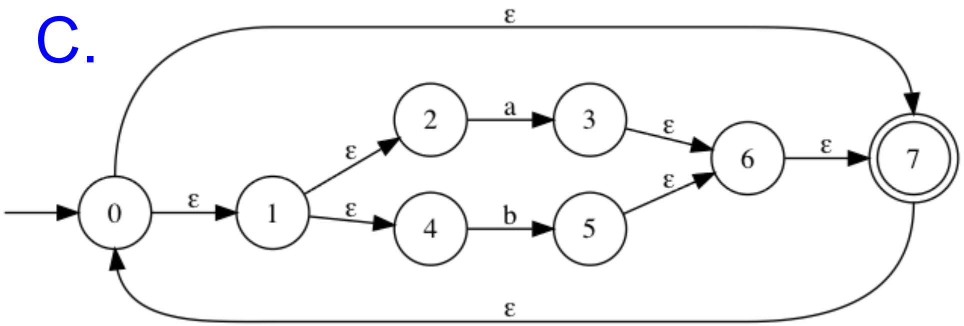

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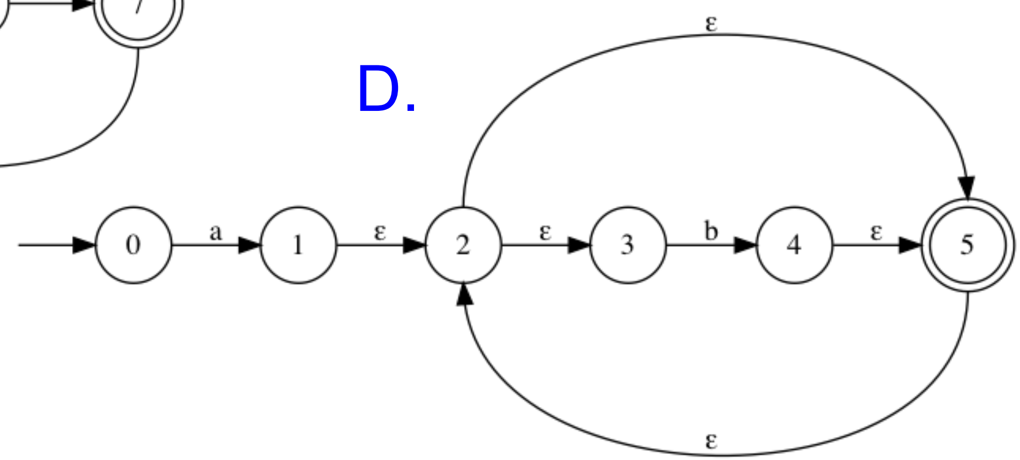

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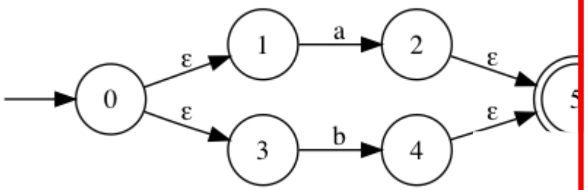

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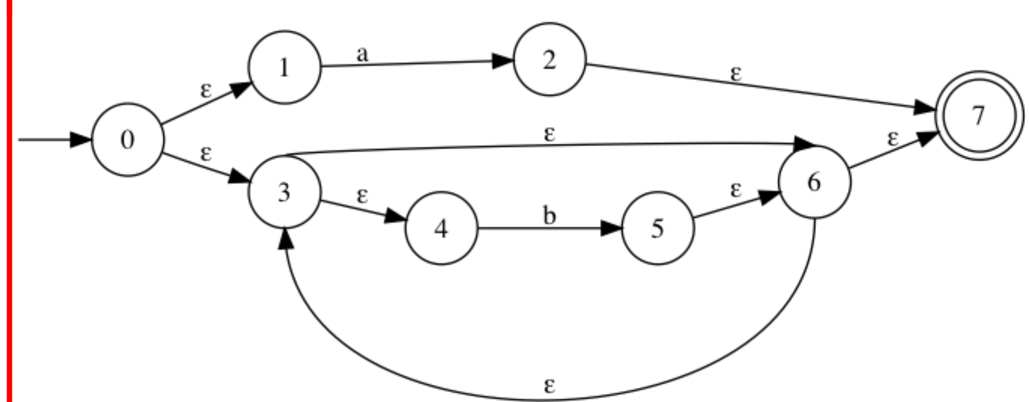
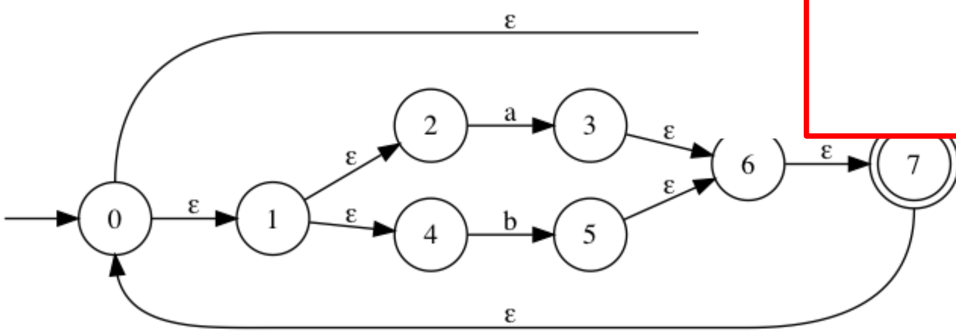
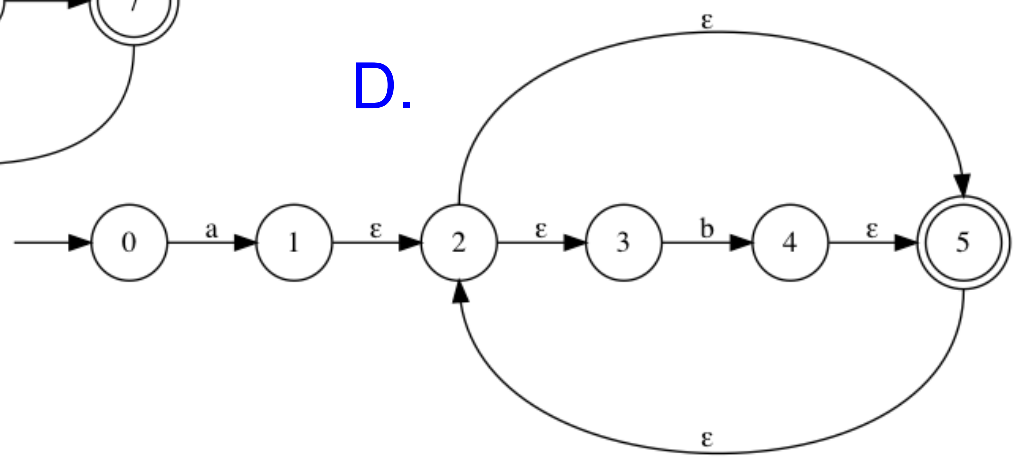




D.



RE → NFA

Draw NFAs for the regular expression (0|1)*110*

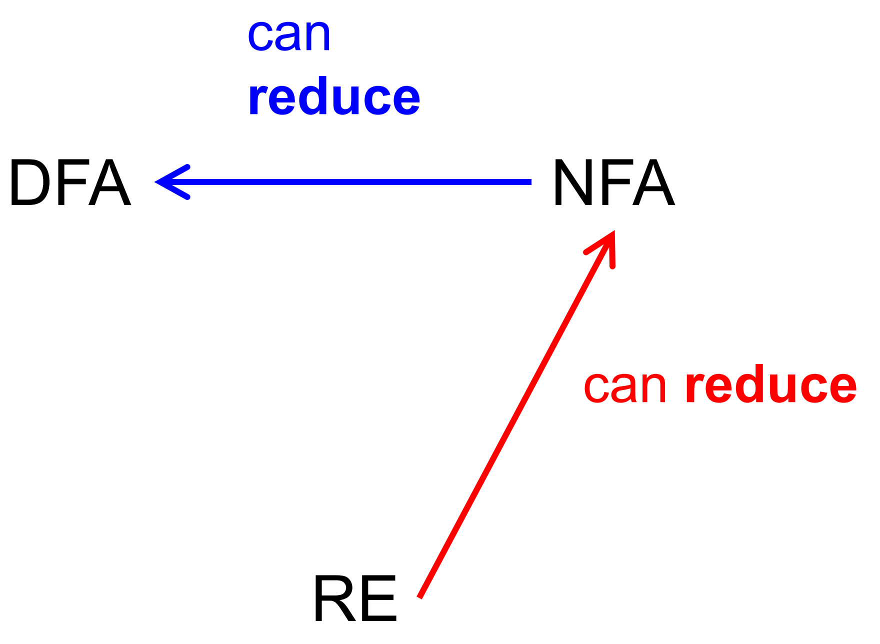
Reduction Complexity

- Given a regular expression *A* of size n...

  Size = # of symbols + # of operations

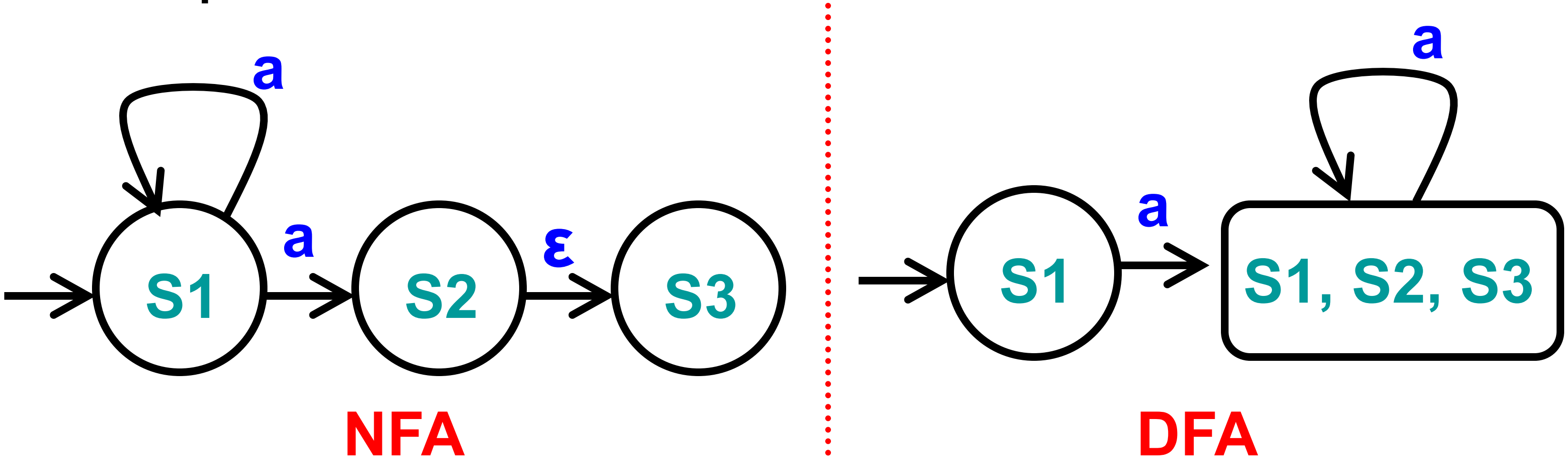
- How many states does <*A*> have?
  - Two added for each **|**, two added for each *****
  - O(n)
  - That's pretty good!

Reducing NFA to DFA



Reducing NFA to DFA

- NFA may be reduced to DFA
  - By explicitly tracking the set of NFA states
- Intuition
  - Build DFA where
    - Each DFA state represents a set of NFA “current states”
- Example



Algorithm for Reducing NFA to DFA

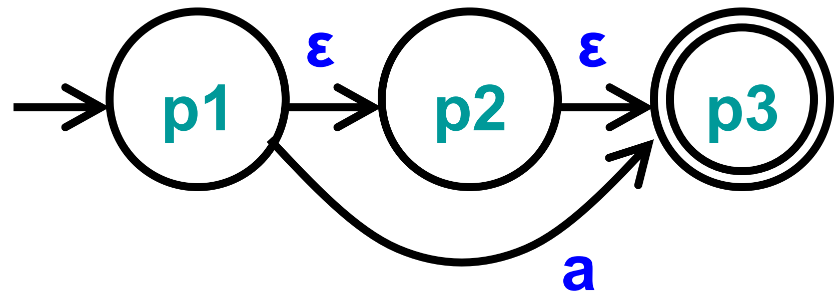
- Reduction applied using the subset algorithm
  - DFA state is a subset of set of all NFA states
- Algorithm
  - Input
    - NFA (Σ, Q, q_0, F_n, δ)
  - Output
    - DFA (Σ, R, r_0, F_d, δ)
  - Using two subroutines
    - ε-closure(δ, p) (and ε-closure(δ, Q))
    - move(δ, p, σ) (and move(δ, Q, σ))
      - (where p is an NFA state)

ε-transitions and ε-closure

- We say $p \xrightarrow{\varepsilon} q$
  - If it is possible to go from state p to state q by taking only ε-transitions in δ
  - If $\exists$ $p, p_1, p_2, \ldots p_n, q \in Q$ such that
    - $\{p,\varepsilon,p_1\} \in \delta$, $\{p_1,\varepsilon,p_2\} \in \delta$, … , $\{p_n,\varepsilon,q\} \in \delta$
- ε-closure(δ, p)
  - Set of states reachable from p using ε-transitions alone
    - Set of states q such that $p \xrightarrow{\varepsilon} q$ according to δ
    - ε-closure(δ, p) = $\{q \mid p \xrightarrow{\varepsilon} q \text{ in } \delta\}$
    - ε-closure(δ, Q) = $\{ q \mid p \in Q, p \xrightarrow{\varepsilon} q \text{ in } \delta \}$
  - Notes
    - ε-closure(δ, p) always includes p
    - We write ε-closure(p) or ε-closure(Q) when δ is clear from context

ε-closure: Example 1

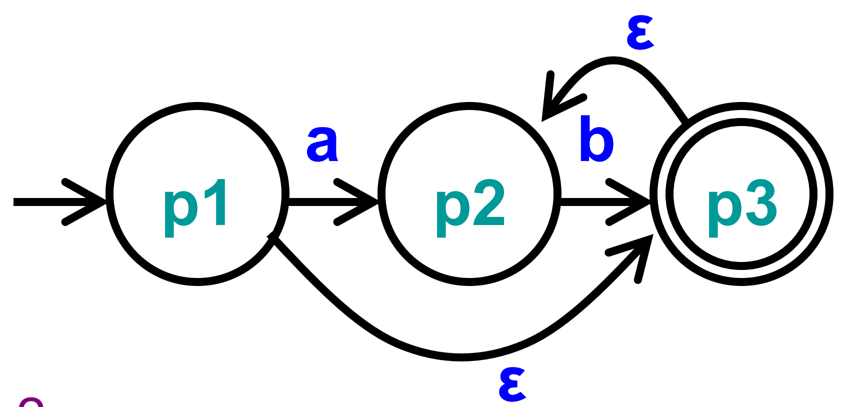
- Following NFA contains
  - $p1 \xrightarrow{\varepsilon} p2$
  - $p2 \xrightarrow{\varepsilon} p3$
  - $p1 \xrightarrow{\varepsilon} p3$
    - Since $p1 \xrightarrow{\varepsilon} p2$ and $p2 \xrightarrow{\varepsilon} p3$



- ε-closures
  - ε-closure(p1) = { p1, p2, p3 }
  - ε-closure(p2) = { p2, p3 }
  - ε-closure(p3) = { p3 }
  - ε-closure({ p1, p2 }) = { p1, p2, p3 } ∪ { p2, p3 }

ε-closure: Example 2

- Following NFA contains
  - p1 $\xrightarrow{\varepsilon}$ p3
  - p3 $\xrightarrow{\varepsilon}$ p2
  - p1 $\xrightarrow{\varepsilon}$ p2
    - Since p1 $\xrightarrow{\varepsilon}$ p3 and p3 $\xrightarrow{\varepsilon}$ p2



- ε-closures
  - ε-closure(p1) = { p1, p2, p3 }
  - ε-closure(p2) = { p2 }
  - ε-closure(p3) = { p2, p3 }
  - ε-closure({ p2,p3 }) = { p2 } ∪ { p2, p3 }

ε-closure Algorithm: Approach

- Input: NFA $(\Sigma, Q, q_0, F_n, \delta)$, State Set R
- Output: State Set R'
- Algorithm

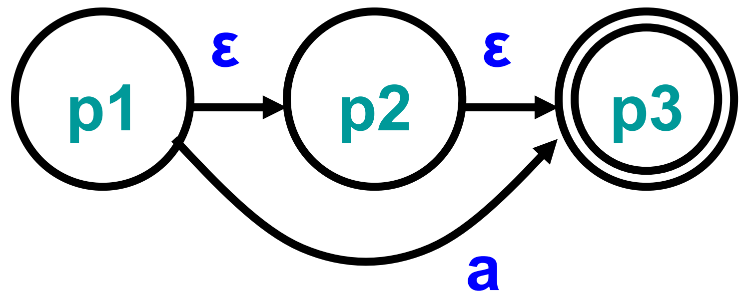
```
Let R' = R                                      // start states
Repeat
    Let R = R'                                  // continue from previous
    Let R' = R ∪ {q | p ∈ R, (p, ε, q) ∈ δ}     // new ε-reachable states
Until R = R'                                    // stop when no new states
```

This algorithm computes a fixed point

ε-closure Algorithm Example

- Calculate ε-closure(δ,{p1})

R	R’
{p1}	{p1}
{p1}	{p1, p2}
{p1, p2}	{p1, p2, p3}
{p1, p2, p3}	{p1, p2, p3}



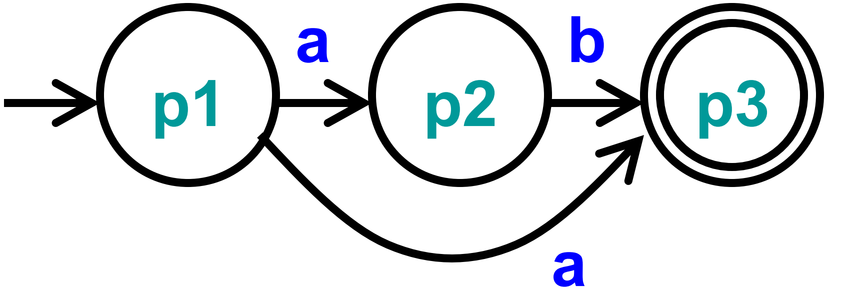
Let R’ = R
Repeat
  Let R= R’
  Let R’ = R $\cup$ {q | p $\in$ R, (p, ε, q) $\in$ δ}
Until R = R’

Calculating move(p,σ)

- move(δ,p,σ)
  - Set of states reachable from p using exactly one transition on symbol σ
    - Set of states q such that {p, σ, q} ∈ δ
    - move(δ,p,σ) = { q | {p, σ, q} ∈ δ }
    - move(δ,Q,σ) = { q | p ∈ Q, {p, σ, q} ∈ δ }
      - i.e., can "lift" move() to a *set* of states Q
  - Notes:
    - move(δ,p,σ) is Ø if no transition (p,σ,q) ∈ δ, for any q
    - We write move(p,σ) or move(R,σ) when δ clear from context

move(p,σ) : Example 1

- Following NFA
  - Σ = { a, b }

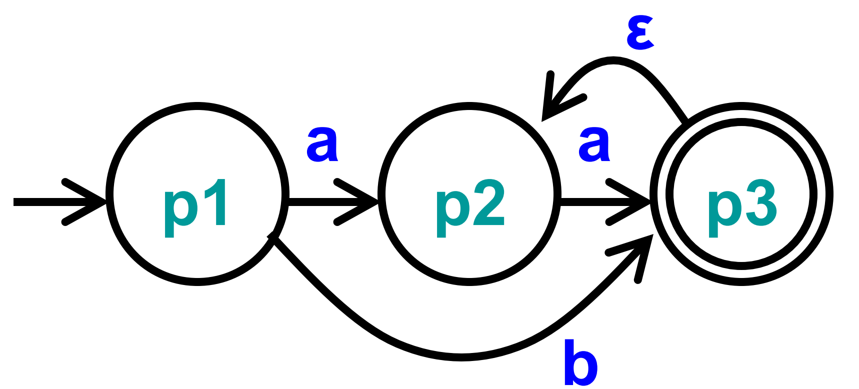


- Move
  - move(p1, a) = { p2, p3 }
  - move(p1, b) = Ø
  - move(p2, a) = Ø
  - move(p2, b) = { p3 }
  - move(p3, a) = Ø
  - move(p3, b) = Ø

move({p1,p2},b) = { p3 }

move(p,σ) : Example 2

- Following NFA
  - Σ = { a, b }



- Move
  - move(p1, a) = { p2 }
  - move(p1, b) = { p3 }
  - move(p2, a) = { p3 }
  - move(p2, b) = Ø
  - move(p3, a) = Ø
  - move(p3, b) = Ø

move({p1,p2},a) = {p2,p3}

NFA $\rightarrow$ DFA Reduction Algorithm ("subset")

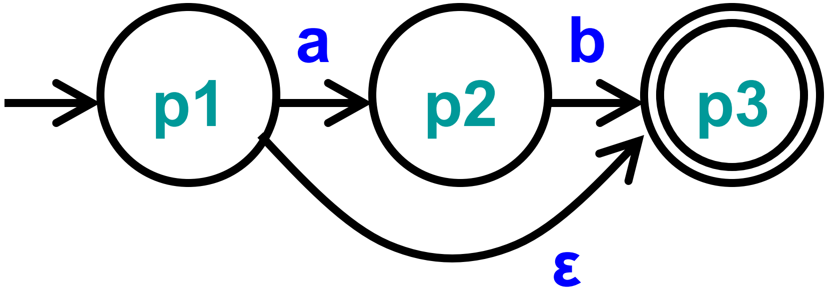
- Input NFA $(\Sigma, Q, q_0, F_n, \delta)$, Output DFA $(\Sigma, R, r_0, F_d, \delta')$
- Algorithm

Let r_0 = ε-closure(δ,q_0), add it to R	// DFA start state
While ∃ an unmarked state r ∈ R	// process DFA state r
Mark r	// each state visited once
For each σ ∈ Σ	// for each symbol σ
Let E = move(δ,r,σ)	// states reached via σ
Let e = ε-closure(δ,E)	// states reached via ε
If e ∉ R	// if state e is new
Let R = R ∪ {e}	// add e to R (unmarked)
Let δ' = δ' ∪ {r, σ, e}	// add transition r→e on σ
Let $F_d = \{r \mid \exists\, s \in r \text{ with } s \in F_n\}$	// final if include state in F_n

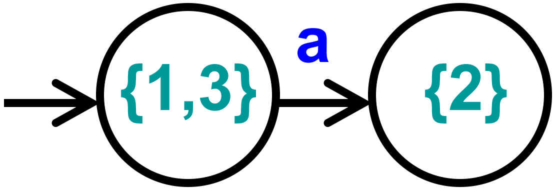
NFA → DFA Example 1

- Start = ε-closure(δ,p1) = { {p1,p3} }
- R = { {p1,p3} }
- r ∈ R = {p1,p3}
- move(δ,{p1,p3},a) = {p2}
  - e = ε-closure(δ,{p2}) = {p2}
  - R = R ∪ {{p2}} = { {p1,p3}, {p2} }
  - δ' = δ' ∪ {{p1,p3}, a, {p2}}
- move(δ,{p1,p3},b) = Ø

NFA



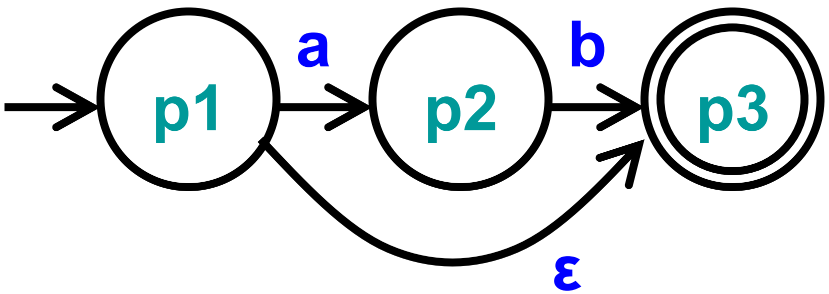
DFA



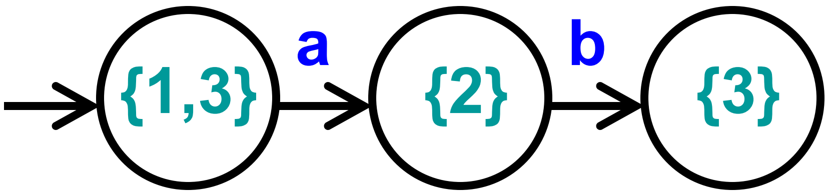
NFA → DFA Example 1 (cont.)

- R = { {p1,p3}, {p2} }
- r ∈ R = {p2}
- move(δ,{p2},a) = Ø
- move(δ,{p2},b) = {p3}
  - e = ε-closure(δ,{p3}) = {p3}
  - R = R ∪ {{p3}} = { {p1,p3}, {p2}, {p3} }
  - δ' = δ' ∪ {{p2}, b, {p3}}

NFA

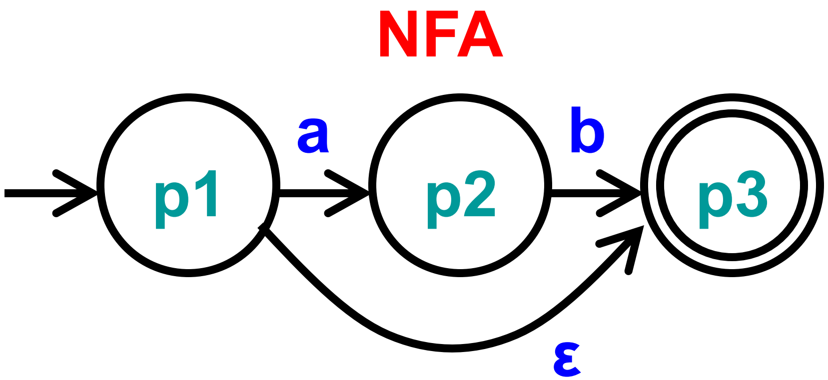


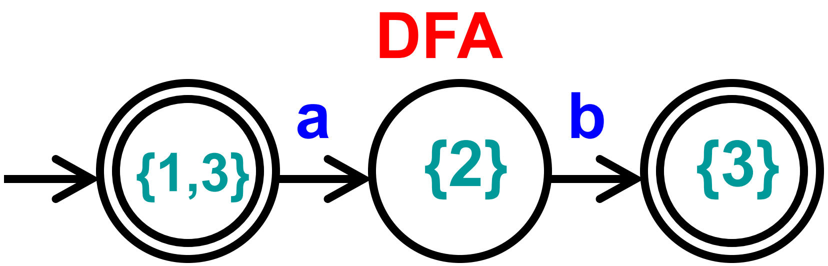
DFA



NFA → DFA Example 1 (cont.)

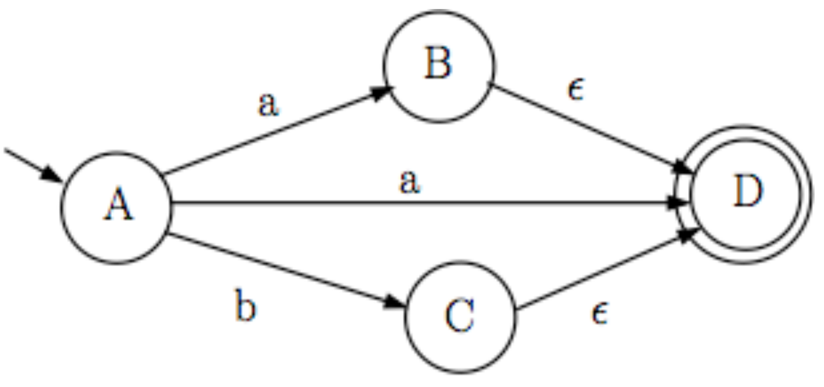
- $R = \{ \{p1,p3\}, \{p2\}, \{p3\} \}$
- $r \in R = \{p3\}$
- Move({p3},a) = Ø
- Move({p3},b) = Ø
- Mark {p3}, exit loop
- $F_d = \{\{p1,p3\}, \{p3\}\}$
  - Since $p3 \in F_n$
- Done!



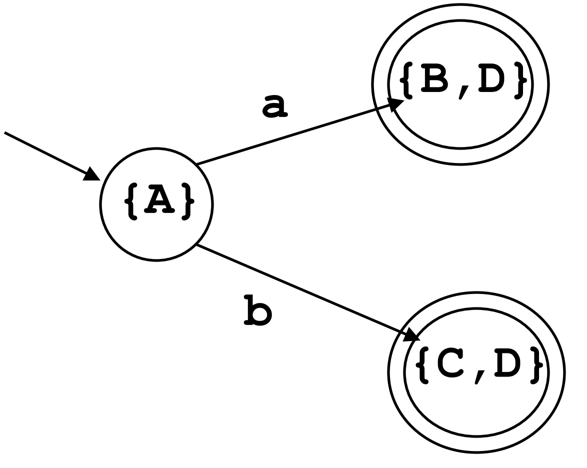


NFA → DFA Example 2

- NFA

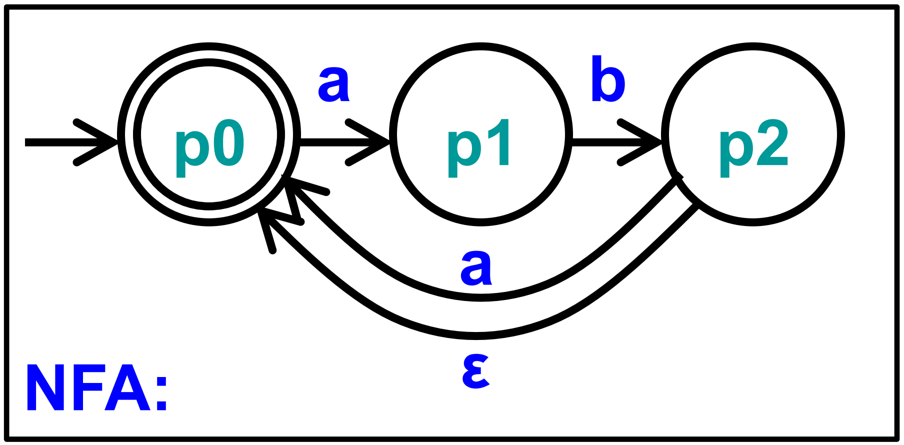


- DFA

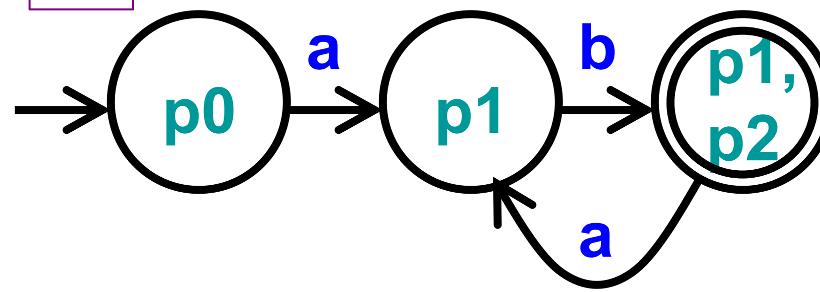


R = { {A}, {B,D}, {C,D} }

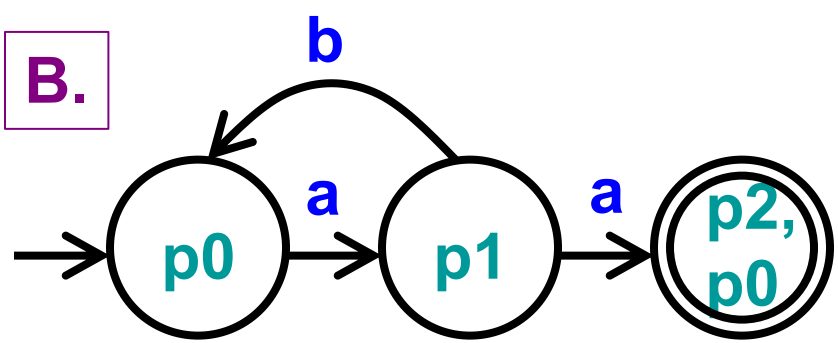
Quiz 4: Which DFA is equiv to this NFA?



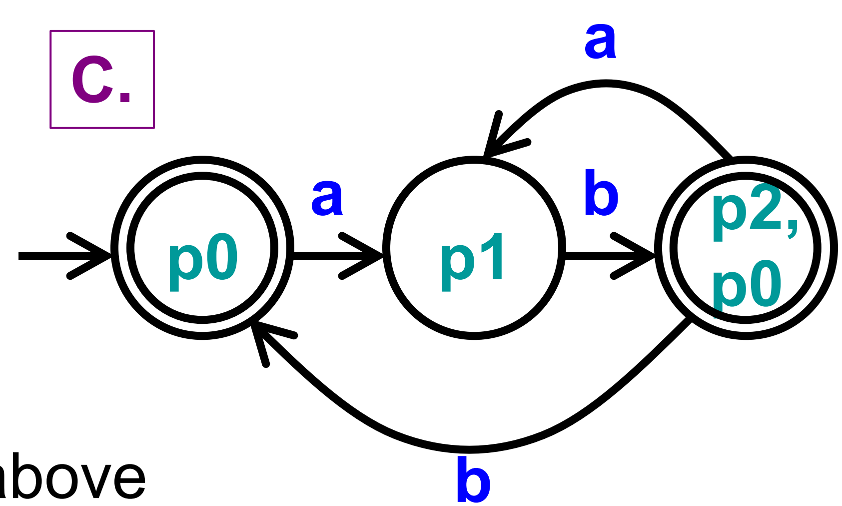
A.



B.

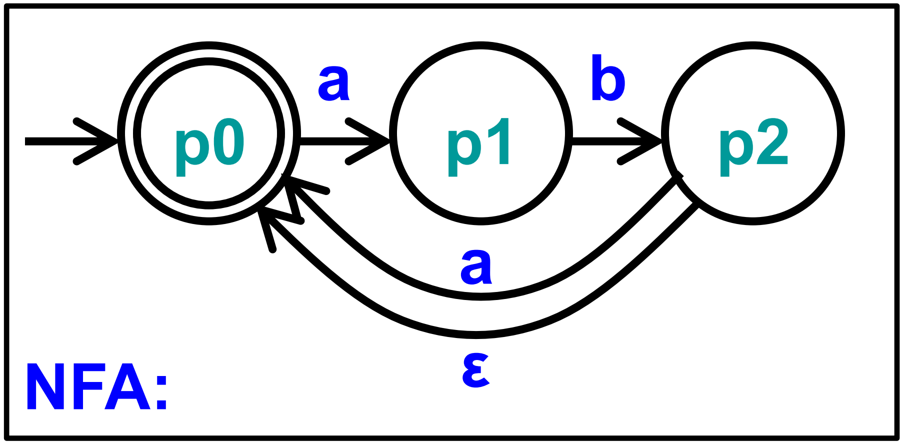


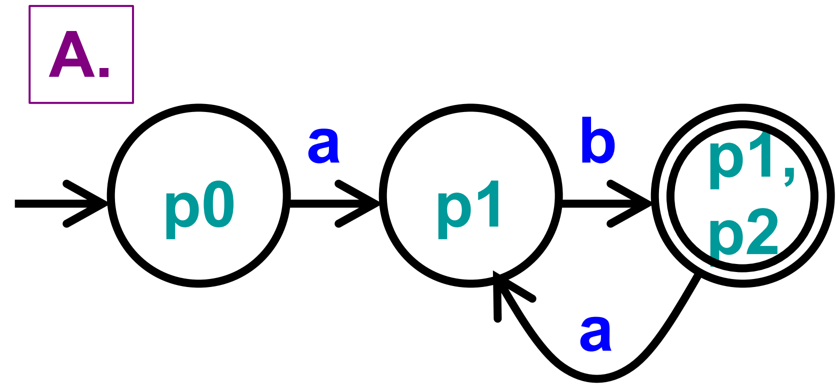
C.

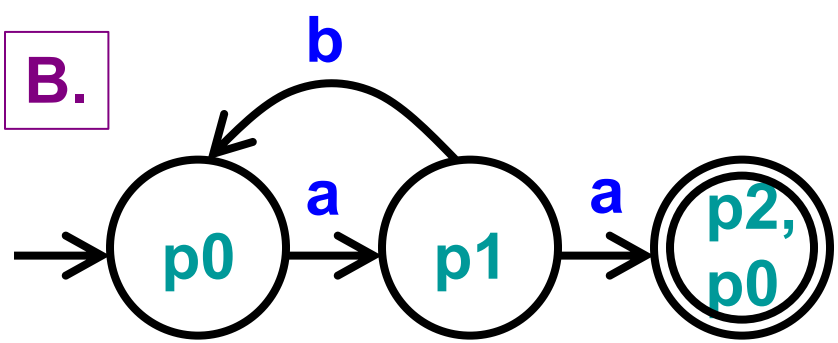


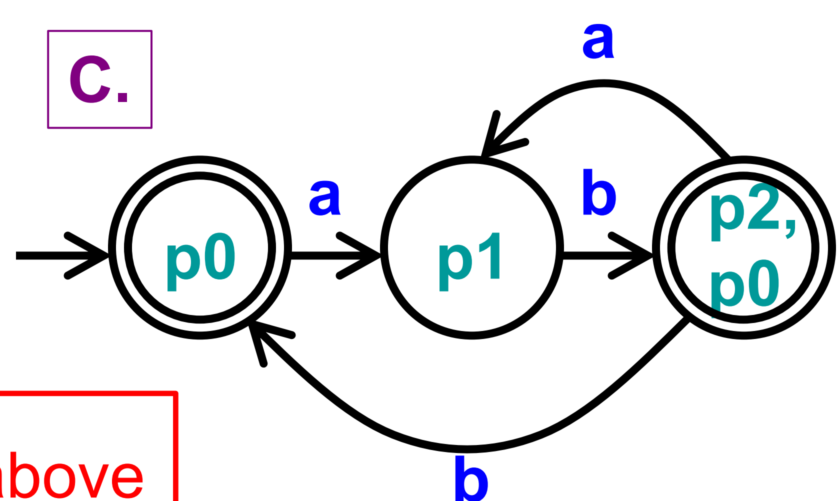
D. None of the above

Quiz 4: Which DFA is equiv to this NFA?



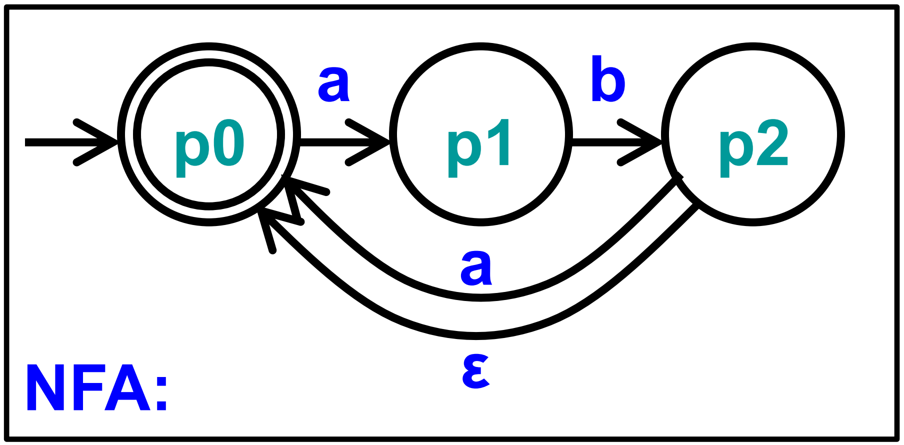


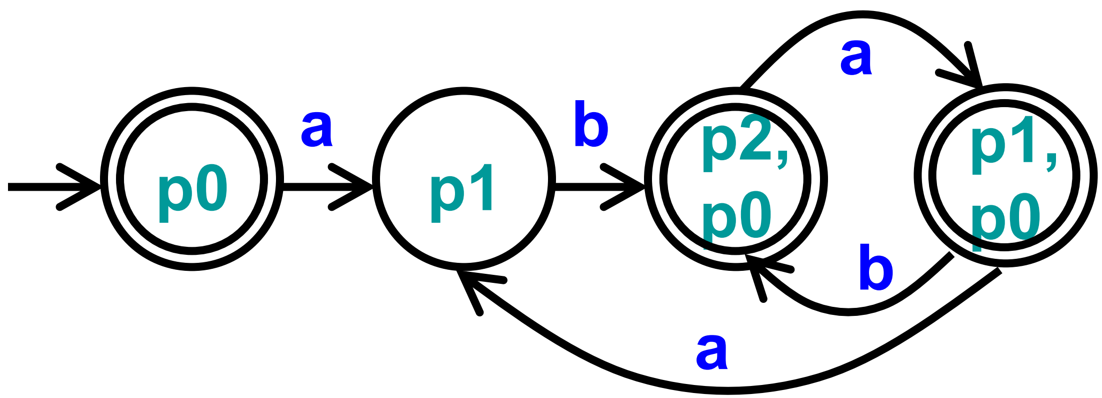




D. None of the above

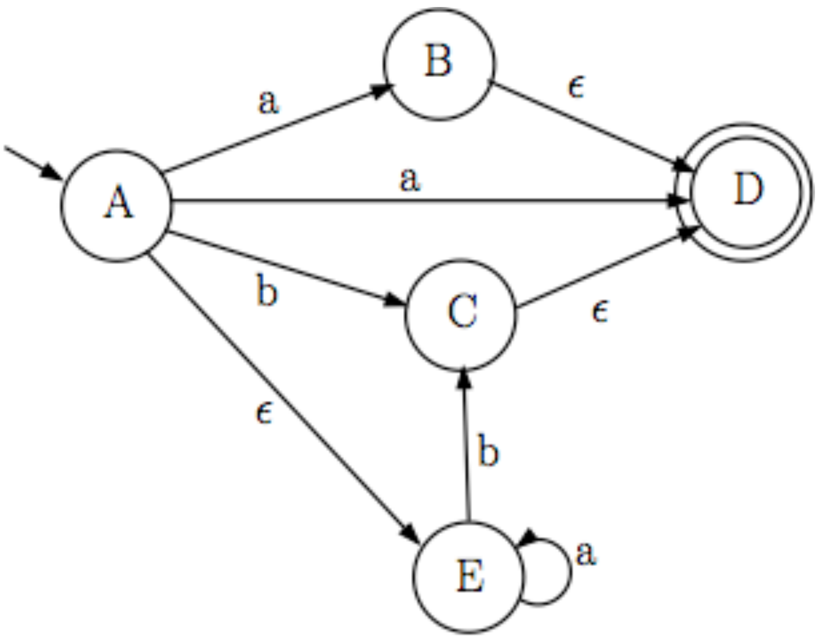
Actual Answer



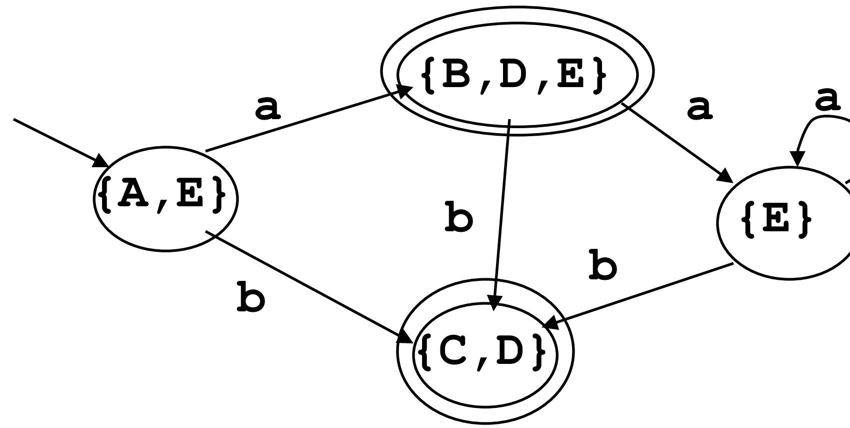


NFA → DFA Example 3

- NFA

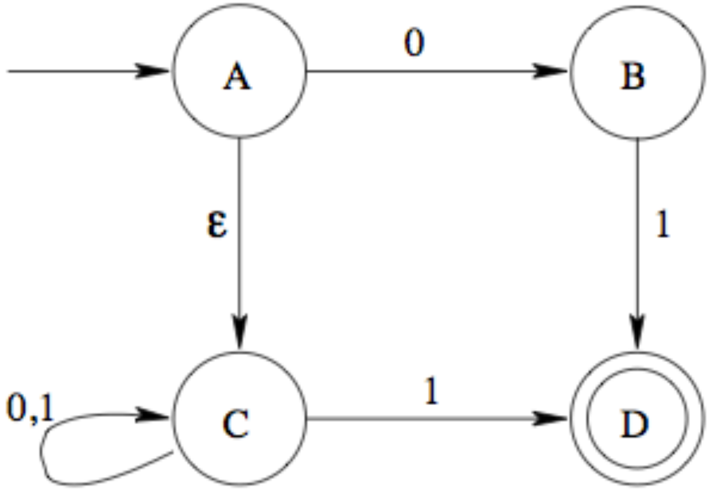


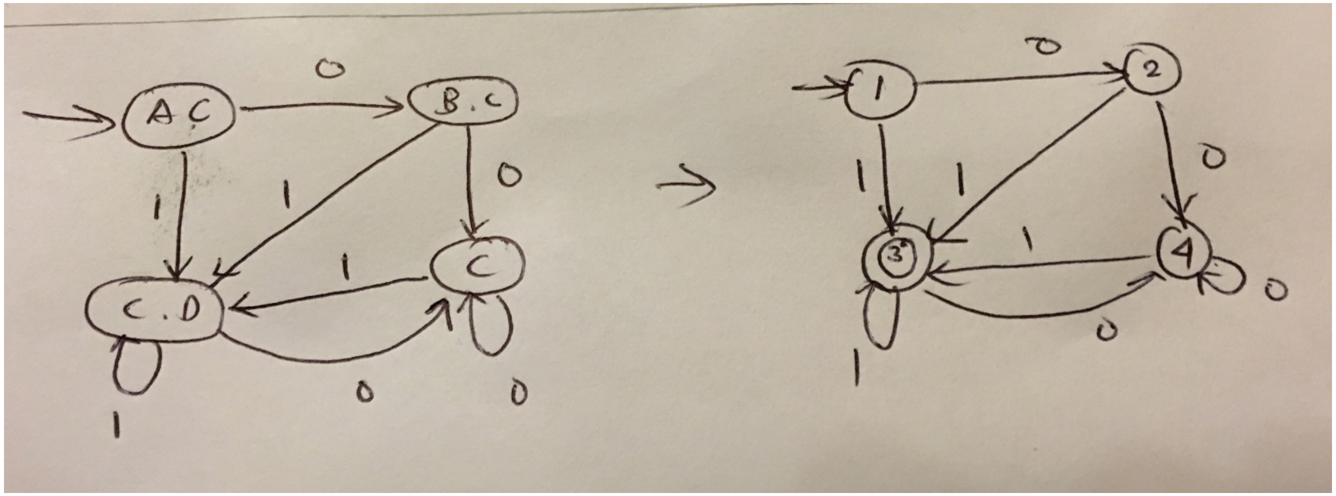
- DFA



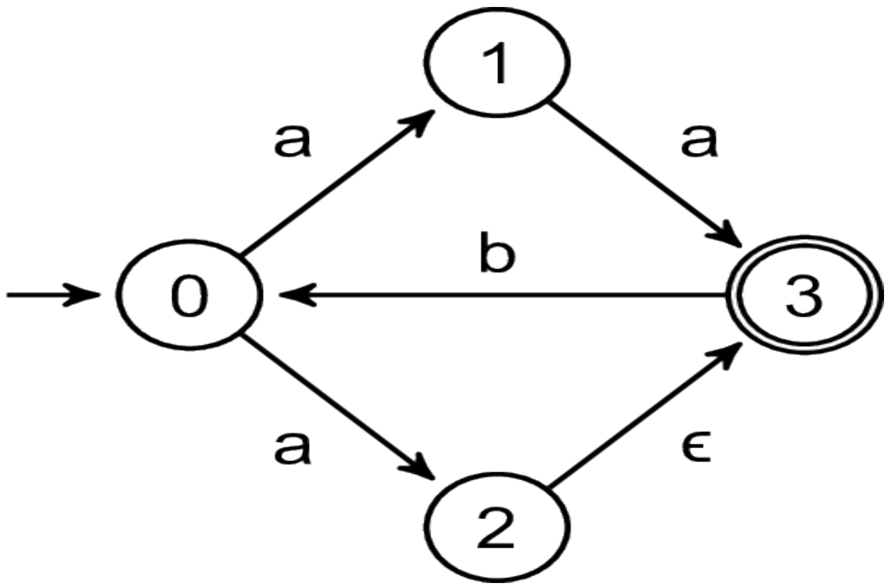
R = { {A,E}, {B,D,E}, {C,D}, {E} }

NFA → DFA Example





NFA → DFA Practice



NFA → DFA Practice