Introduction to Data Structures

- Elements of data structures
- Our approach
- Short review of asymptotics

Course Overview:

- Fundamental data structures + algorithms
- Mathematical techniques for analyzing them
- Implementation

Data structures are FUNDAMENTAL!

- All fields of CS involve storing, retrieving and processing data

- Information retrieval
- Geographic Inf. Systems
- Machine Learning
- Text/String processing
- Computer Graphics
- ....

Basic Elements in Study of data structures

- **Modeling**: How real world objects are encoded
- **Operations**: Allowed functions to access + modify structure
- **Representation**: Mapping to memory
- **Algorithms**: How are operations performed?

Our approach:

- **Theoretical**: Algorithms + Asymptotic Analysis
- **Practical**: Implementation + practical efficiency

Asymptotic Analysis:

- Run time as function of n: no. of items
- Worst-case, average case, randomized,...
- **Amortized** - average over series of ops.

Asymptotic: "Big-O"

- Ignore constants
- Focus on large n

$T(n) = 34n^2 + 15n\log n + 143$

$T(n) = O(n^2)$

Common:

$O(1)$: constant time ☺ [Hashmap]

$O(\log n)$: log-time (good) [Binary search]

$O(n^p)$: p= constant: poly time

$O(\sqrt{n})$

Basic Data Structures I

- ADTs
- Lists, Stacks, Queues
- Sequential Allocation

Abstract Data Type (ADT)

- Abstracts the functional elements of a data structure (math) from its implementation (algorithm / programming)

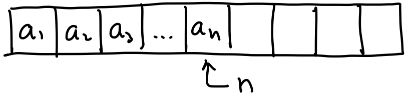
Linear List ADT:

Stores a sequence of elements $\langle a_1, a_2, \ldots, a_n \rangle$. Operations:

- **init()** - create an empty list
- **get(i)** - returns a_i
- **set(i,x)** - sets i^{th} element to x
- **insert(i,x)** - inserts x prior to i^{th} (moving others back)
- **delete(i)** - deletes i^{th} item (moving others up)
- **length()** - returns num. of items

Implementations:

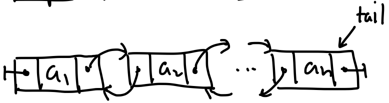
Sequential: Store items in an array



Linked allocation: linked list

Singly: head →

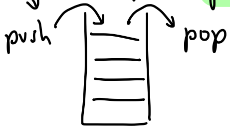
Doubly: head →

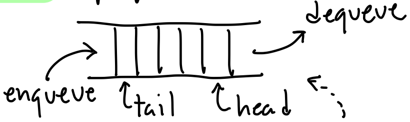


Performance varies with implementation

Stack: All access from one side

(top) - **push** + **pop**





Queue: FIFO list: **enqueue** inserts at **tail** and **dequeue** deletes from **head**

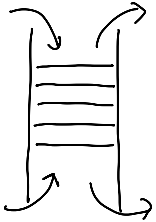
Doubling Reallocation:

When array of size n overflows

- allocate new array size 2n
- copy old to new
- remove old array

Dynamic Lists + Sequential Allocation: What to do when your array runs out of space?

Deque ("deck"): Can insert or delete from either end



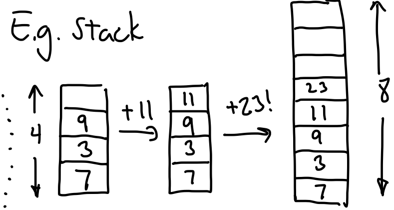
Basic Data Structures II

- Amortized analysis of dynamic stack

Dynamic (Sequential) Allocation

- When we overflow, double

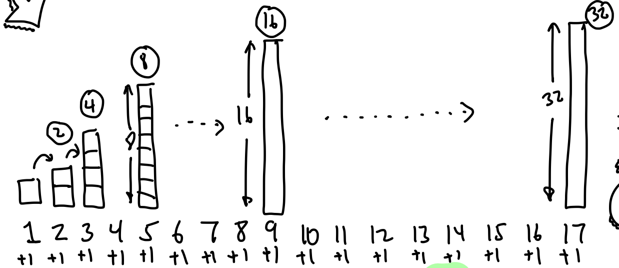
E.g. Stack



Cost model (Actual cost)

Cheap: No reallocation → 1 unit

Expensive: Array of size n is reallocated to size 2n } 2n+1



Total = 17+(2+4+8+16+32)= 79

Amortized Cost:

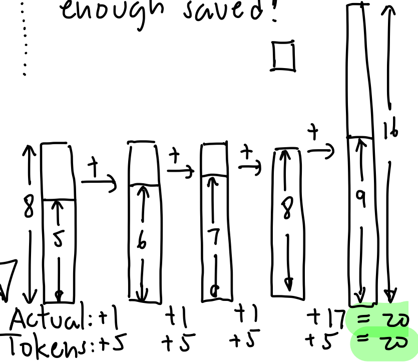
Starting from an empty structure, suppose that any sequence of m ops takes time T(m). The amortized cost is T(m)/m.

Thm: Starting from an empty stack, the amortized cost of our stack operations is at most 5.

[i.e. any seq. of m ops has cost ≤ 5·m]

Charging Argument:

- Each request of push/pop we charge user 5 "work tokens"
- We use 1 token to pay for the operation + put other 4 in bank account.
- Will show there is enough in bank account to pay actual costs.



Proof:

- Break the full sequence after each reallocation → run

1 2 | 3 | 4 5 | 6 7 8 9 | 10 11 … 16 17 |

- At start of a run there are n+1 items in stack and array size is 2n
- There are at least n ops before the end of run
- During this time we collect at least 5n tokens
  - → 1 for each op
  - → 4 for deposit
- Next reallocation costs 4n, but we have enough saved!

□

Basic Data Structures III

- Dynamic Stack - Wrap-up
- Multilists + Sparse Matrices

Dynamic Stack:

- Showed doubling ⇒ Amortized $O(1)$
- Other strategies?

Fixed Increment: Increase by a fixed constant

$n \longrightarrow n+100$

Fixed factor: Increase by a fixed constant factor (not nec. 2)

$n \rightarrow 5 \cdot n$

Squaring: Square the size (or some other power)

$n \rightarrow n^2$ or $n \rightarrow \lceil n^{1.5} \rceil$

Which of these provide $O(1)$ amortized cost per operation?

Leave as exercise ☹

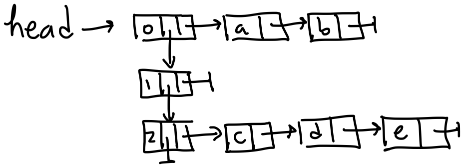
(Spoiler alert!)

Fixed increment → no

Fixed factor → yes

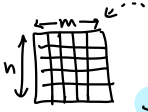
Squaring → yes

Multilists: Lists of lists



Sparse Matrices:

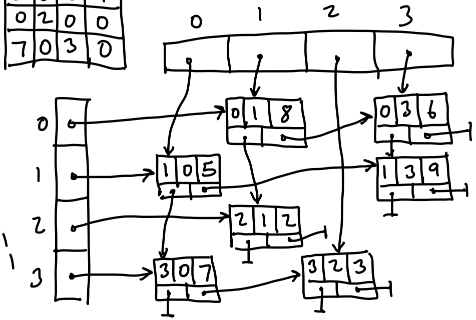
An n x m matrix has n·m entries and takes (naively) $O(n \cdot m)$ space



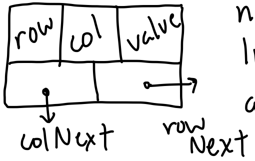
Sparse matrix: Most entries are zero

Idea: Store only nonzero entries linked by row and column

$$\begin{bmatrix} 0 & 8 & 0 & 6 \\ 5 & 0 & 0 & 9 \\ 0 & 2 & 0 & 0 \\ 7 & 0 & 3 & 0 \end{bmatrix}$$

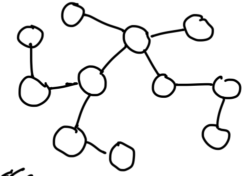


Node:

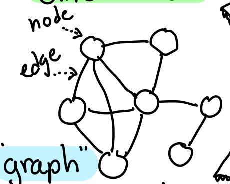


Trees: Basic Concepts and Definitions

Tree (or "**Free Tree**")
- undirected
- connected
- acyclic graph

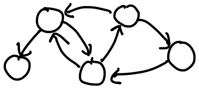


Undirected



"graph"

Directed

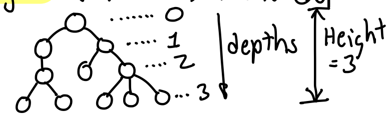


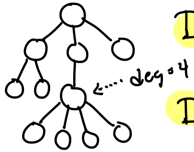
"digraph"

Graph: $G=(V,E)$
V = finite set of vertices (nodes)
E = set of edges (pairs of vertices)

Depth: path length from root

Height: (of tree) max depth

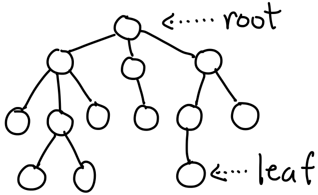




Degree (of node): number of children

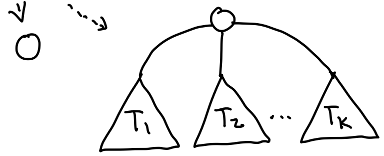
Degree (of tree): max. degree of any node

Rooted tree: A free tree with root node

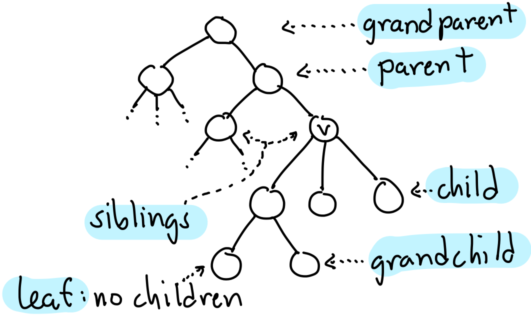


Formal definition:

Rooted tree: is either
- single node (root)
- set of one or more rooted trees ("subtrees") joined to a common root



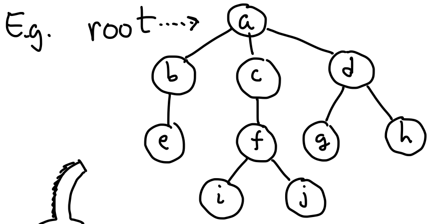
"Family" Relations

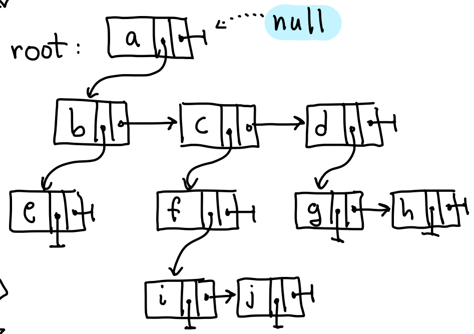


Trees Representation + Binary Trees (1)

Representing rooted trees:

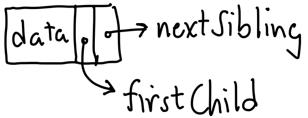
Each node stores a (linked) list of its children



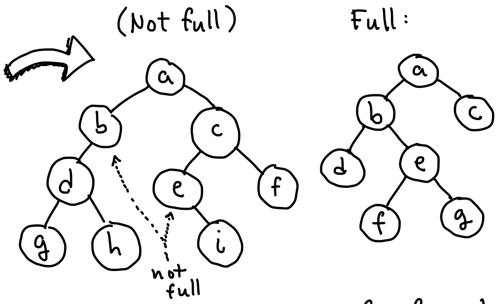


called the **Binary representation**

Node structure:



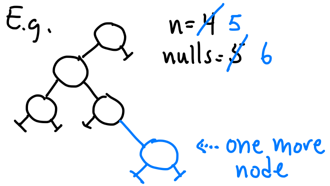
Binary tree: A rooted tree of degree 2, where each node has two children (possibly null) left + right



Full: Every non-leaf node has 2 children

Wasted space?

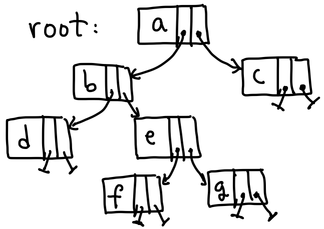
Theorem: A binary tree with n nodes has n+1 null links



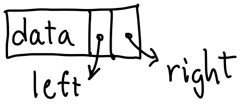
In Java:

```
class BTNode<E> {
    E data;
    BTNode<E> left;
    BTNode<E> right;
    ....
}
```

generic data



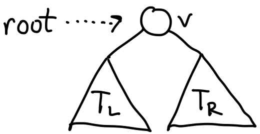
Node structure:



Binary Trees: Traversals, Extension, and More

Traversals: How to (systematically) visit the nodes of a rooted tree?

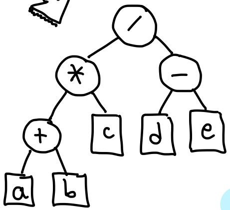
Binary Tree Traversals (can be generalized)



- process/visit v
- traverse T_L
- traverse T_R

} recursive

```
traverse(BTNode v) {
    if (v == null) return;
    visit/process v      ← Preorder
    traverse(v.left)
    visit/process v      ← Inorder
    traverse(v.right)
    visit/process v      ← Postorder
}
```



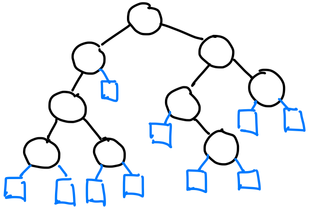
Preorder: / * + a b c - d e

Postorder: a b + c * d e - /

Inorder: a + b * c / d - e

Those wasteful null links....

Extended binary tree: Replace each null link with a special leaf node: **external node**



Thm: An extended binary tree with n internal nodes (black) has n+1 external nodes (blue)

Observation: Every extended binary tree is **full**

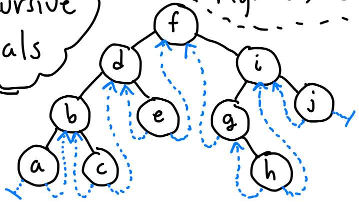
Another way to save space...

Threaded binary tree: Store (useful) links in the null links. (Use a mark bit to distinguish link types.)

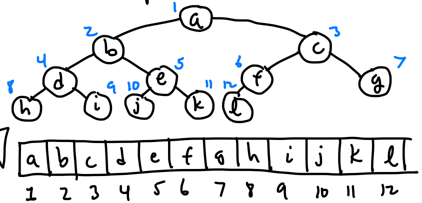
E.g. **Inorder Threads:**

Null left → inorder predecessor

Null right → " successor



Complete Binary Tree: All levels full (except last)



$\text{parent}(i) = \lfloor i/2 \rfloor$

$\text{left}(i) = 2 \cdot i$

$\text{right}(i) = 2 \cdot i + 1$

Challenge: Nonrecursive traversals

Binary Search Trees I

- Basic definitions
- Finding keys

Search: Given a set of n **entries** each associated with **key** x_i and **value** v_i

- Store for quick access + updates
- **Ordered**: Assume that keys are totally ordered: <, >, ==

Dictionary:

insert(Key x, Value v)
- insert (x, v) in dict. (No duplicates)

delete(Key x)
- delete x from dict. (Error if x not there)

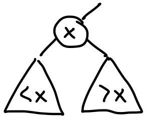
find(Key x)
- returns a reference to associated value v, or **null** if not there.

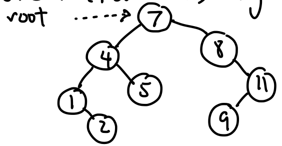
Sequential Allocation?

- Store in array sorted by key
  - → **Find**: O(log n) by binary search
  - → **Insert/Delete**: O(n) time

Can we achieve O(log n) time for all ops? ☆ **Binary Search Trees** ☆

Idea: Store entries in binary tree sorted (inorder traversal) by key





Find: How to find a key in the tree?

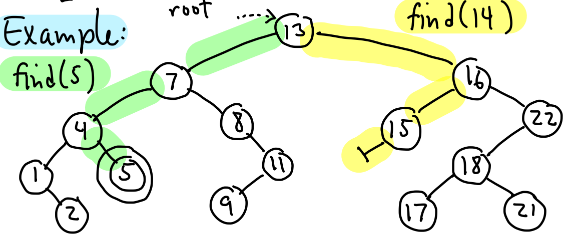
- Start at root p ← root
- if (x < p.key) search left
- if (x > p.key) search right
- if (x == p.key) found it!
- if (p == null) not there!

```
Value find(Key x, BSTNode p){
    if (p == null) return null
    else if (x < p.key)
        return find(x, p.left)
    else if (x > p.key)
        return find(x, p.right)
    else return p.value
}
```

Example:

find(5)

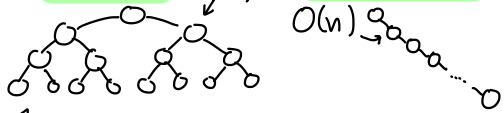
find(14)



Efficiency: Depends on tree's height

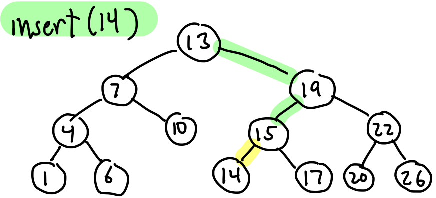
Balanced: O(log n)

Unbalanced: O(n)



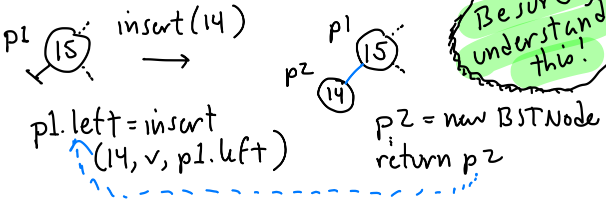
Binary Search Trees II
- insertion
- deletion

Insert (Key x, Value v)
- find x in tree
- if found ⇒ error! duplicate key
- else: create new node where we "fell out"



```
BSTNode insert(Key x, Value v, BSTNode p){
    if (p==null)
        p = new BSTNode(x,v)
    else if (x < p.key)
        p.left = insert(x,v, p.left)
    else if (x > p.key)
        p.right = insert(x,v, p.right)
    else throw exception → Duplicate!
    return p
}
```

Why did we do:
p.left = insert(x,v, p.left)?



p1.left = insert(14, v, p1.left)

p2 = new BSTNode
return p2

Be sure you understand this!

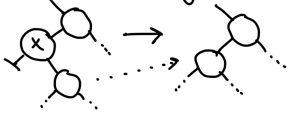
Delete (Key x)
- find x
- if not found → error
- else: remove this node + restore BST structure

How?

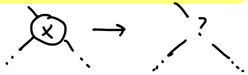
3 cases:

① ⓧ is a leaf

② ⓧ has single child

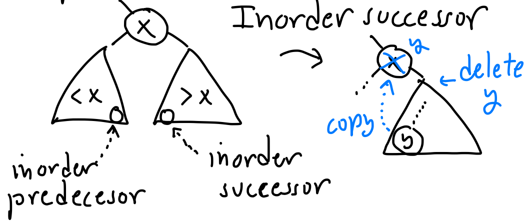


3. ⓧ has two children



Find replacement node ⓨ, copy to ⓧ, and then delete ⓨ

Replacement Node?

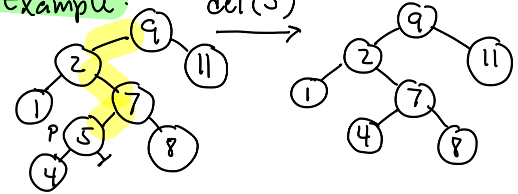


Binary Search Trees III

- deletion
- analysis
- Java

```
BSTNode delete(Key x, BSTNode p){
    if (p == null) ERROR! Key not found
    else
        if (x < p.key)
            p.left = delete(x, p.left)
        else if (x > p.key)
            p.right = delete(x, p.right)
        else if (either p.left or p.right null)
            if (p.left == null)
                return p.right
            if (p.right == null)
                return p.left
        else
            r = findReplacement(p)
            copy r's contents to p
            p.right = delete(r.key, p.right)
    return p
}
```

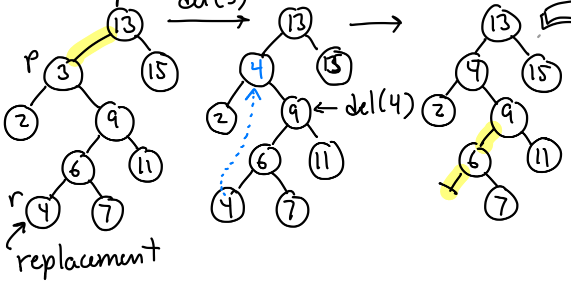
Example:



Find Replacement Node

```
BSTNode findReplacement(BSTNode p){
    BSTNode r = p.right
    while(r.left ≠ null)
        r = r.left
    return r
}
```

Example:



Analysis:

All operations (find, insert, delete) run in $O(h)$ time, where h = tree's height

But height can vary from $O(\log n)$ to $O(n)$...

Expected case is good

Thm: If n keys are inserted in random order, expected height is $O(\log n)$.

Java Implementation:

- Parameterize Key + Value types: extends Comparable
  class BinSearchTree<K,V>{...
- BSTNode - inner class
- Private data: BSTNode root
- insert, delete, find: local
- provide public fns
  insert, delete, find

AVL Trees I

- Basic defs
- Height props
- Rotations

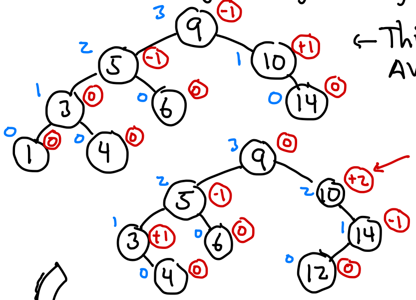
AVL Height Balance

- for each node v, the heights of its subtrees differ by ≤ 1.

AVL tree: A binary search tree that satisfies this condition

Balance factor:

$$bal(v) = hgt(v.right) - hgt(v.left)$$



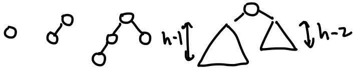
← This is an AVL tree

Not an AVL tree

Does this imply O(log n) height?

Worst cases:

height: $h =$ 0 1 2 .... h



nodes: $n =$ 1 2 4 7 12 20 ...

$n+1 =$ 2 3 5 8 13 21 ...

Recall: $F_0 = 0$, $F_1 = 1$, $F_h = F_{h-1} + F_{h-2}$

Conjecture: Min no. of nodes in AVL tree of height h is $F_{h+3} - 1$

Theorem: An AVL tree of height h has at least $F_{h+3}-1$ nodes.

Proof: (Induct. on h)

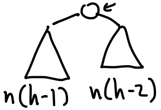
$h=0: \quad n(h) = 1 = F_3 - 1$

$h=1: \quad n(h) = 2 = F_4 - 1$

$h \geq 2: \quad n(h) = 1 + n(h-1) + n(h-2)$

$\quad = 1 + (F_{h+2} - 1) + (F_{h+1} - 1)$

$\quad = (F_{h+2} + F_{h+1}) - 1 = F_{h+3} - 1 \quad \square$



Corollary: An AVL tree with n nodes has height $O(\log n)$

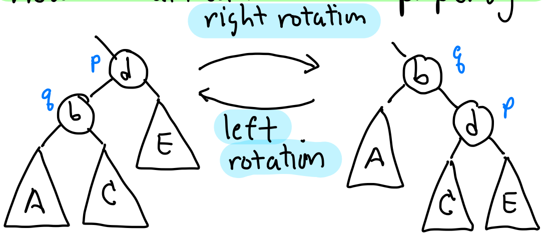
Proof: Fact: $F_h \approx \varphi^h / \sqrt{5}$ where $\varphi = (1+\sqrt{5})/2$ "Golden ratio"

$n \geq \varphi^{h+3} = c \cdot \varphi^h \Rightarrow h \leq \log_\varphi n + c'$

$\Rightarrow h \leq \log_2 n / \log_2 \varphi$

$= O(\log n) \quad \square$

How to maintain the AVL property?



$A < b < C < d < E$ $\quad$ $A < b < C < d < E$

```
BSTNode rotateRight(BSTNode p) {
    BSTNode q = p.left
    p.left = q.right
    q.right = p
    return q
}
```

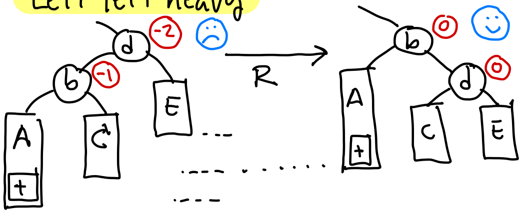
AVL Trees II
- double rotations
- insertion

Find: Same as BST.

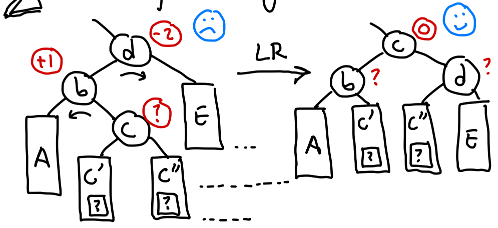
Insert: Same as BST but as we "back out" rebalance

How to rebalance? Bal=-2

Left-left heavy



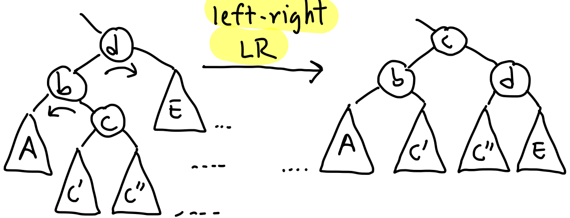
Left-right heavy:

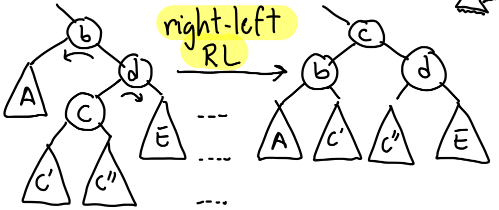


```
AVLNode insert(Key x, Value v, AVLNode p)}
  if (p == null) p = new AVLNode(x,v)
  else if (x < p.key)
      p.left = insert(x,v, p.left)
  else if (x > p.key)
      p.right = insert(x,v,p.right)
  else throw - Error - Duplicate!
  return rebalance(p)
```

```
AVLNode rebalance (AVLNode p)
  if (p==null) return p
  if (balanceFactor(p) < -1)
    if (ht(p.left.left) >= ht(p.left.right))
        p = rotateRight(p)
    else p = rotateLeftRight(p)
  else .... (symmetrical)
  updateHeight(p); return p
```

Double rotations:





```
BSTNode rotateLeftRight (BSTNode p)
  p.left = rotateLeft(p.left)
  return rotateRight(p)
```

AVL Tree:

AVLNode: Same as BSTNode but + member int height

Utilities:

```
int height(AVLNode p)
  return { p == null  -> -1
         { o.w.       -> p.height
void updateheight(AVLNode p)
  p.height = 1 + max(height(p.left),
                     height(p.right))
int balanceFactor(AVLNode p)
  return height(p.right)-
         height(p.left)
```

AVL Trees III

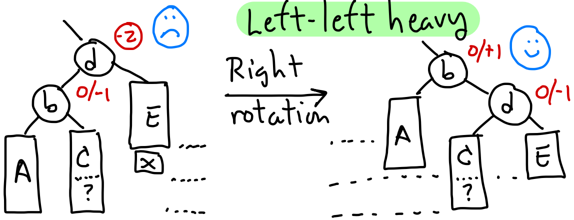
- Deletion
- Examples

Deletion: Basic plan

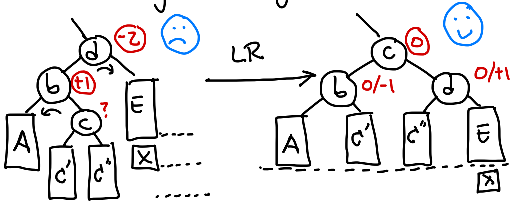
- Apply standard BST deletion
  - find key to delete
  - find replacement node
  - copy contents
  - delete replacement
  - rebalance

Cases: Balance factor -2

Left-left heavy

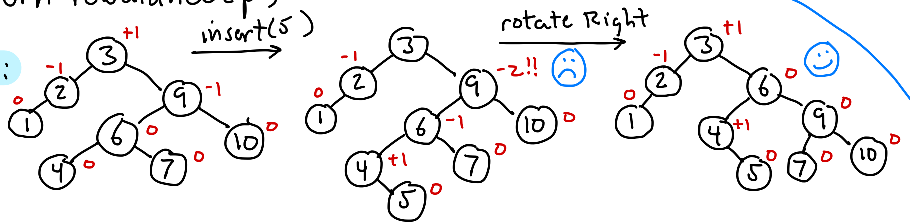


Left-right heavy

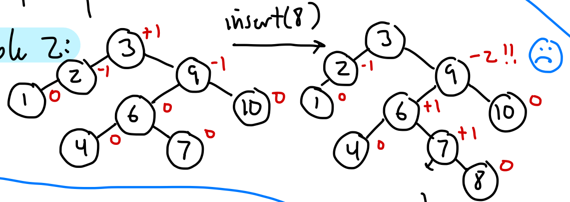


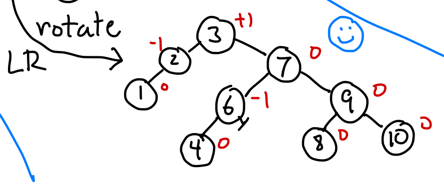
```
AVLNode delete (Key x, AVLNode p)
    :
    same as BST delete
    :
    return rebalance(p)
```

Examples:

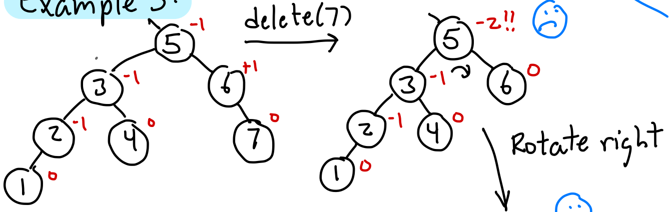


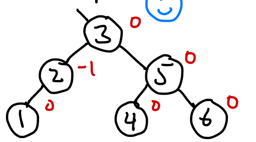
Exampl 2:



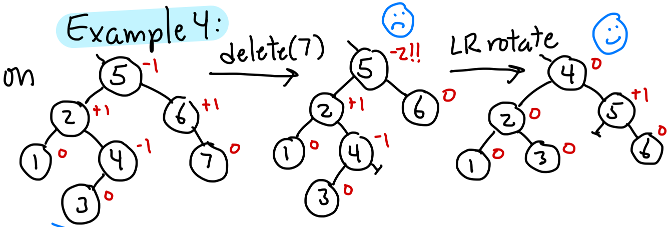


Example 3:

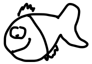




Example 4:



2-3 Trees I



Recap:

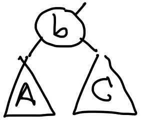
AVL: Height balanced Binary

2-3 tree: Height exact Variable width

Node types:

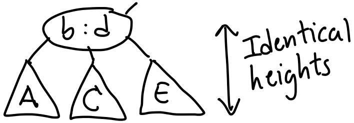
2-Node

1 key
2 children



3-Node

2 keys
3 children

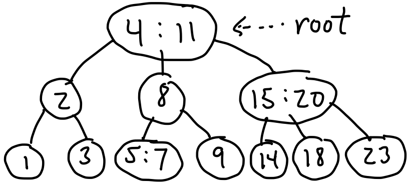


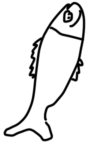
Def: A **2-3 tree** of height h is either:

- Empty ($h=-1$)
- A 2-Node root and two subtrees, each 2-3 tree of height $h-1$
- A 3-Node root and three subtrees... height $h-1$.

Example:

2-3 tree of height 2



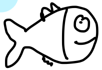


Thm: A 2-3 tree of n nodes has height $O(\log n)$

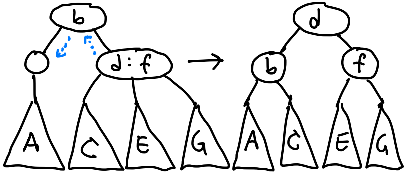
Roughly: $\log_3 n \leq h \leq \log_2 n$

How to maintain balance?

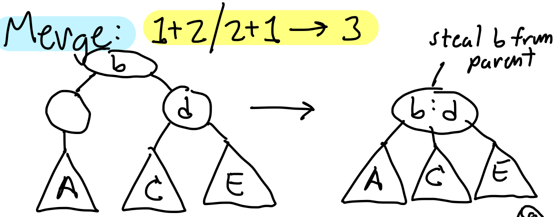
- Split
- Merge
- Adoption (Key rotation)



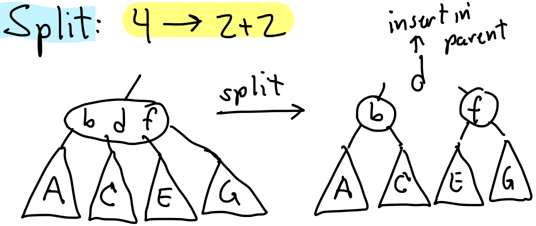
Adoption (Key-Rotation) $1+3 = 2+2$



Merge: $1+2/2+1 \rightarrow 3$

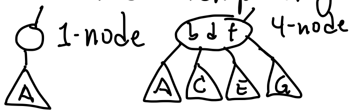


Split: $4 \rightarrow 2+2$



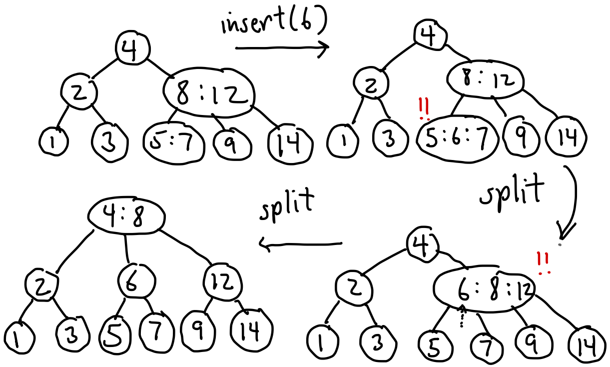
Conceptual tool:

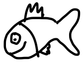
We'll allow 1-nodes + 4-nodes temporary



2-3 Trees II

Insertion example:





Dictionary operations:

Find - straightforward

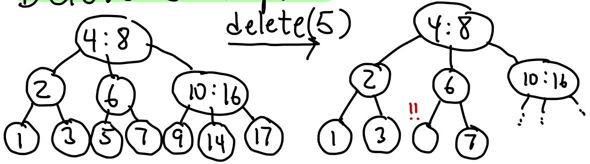
Insert - find leaf node where key "belongs" + add it (may split)

Delete - find / replacement / merge or adopt

Implementation?

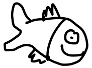
```
class TwoThreeNode {
    int nChildren
    TwoThreeNode children[3]
    Key key[2]
}
```

Delete Example:

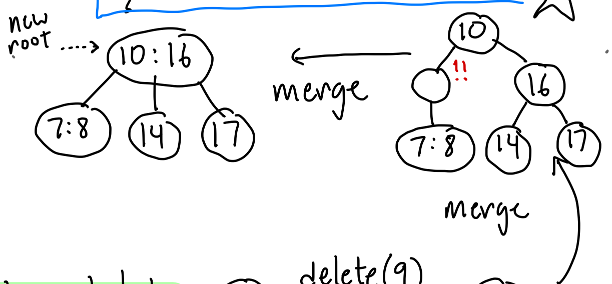


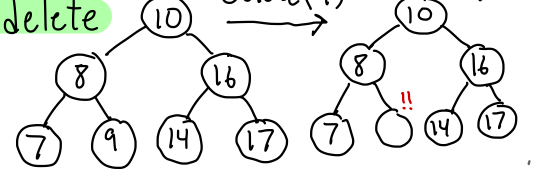
Deletion remedy:

- Have a 3-node neighboring sibling → adopt
- O.w.: Merge with either siblings + steal key from parent

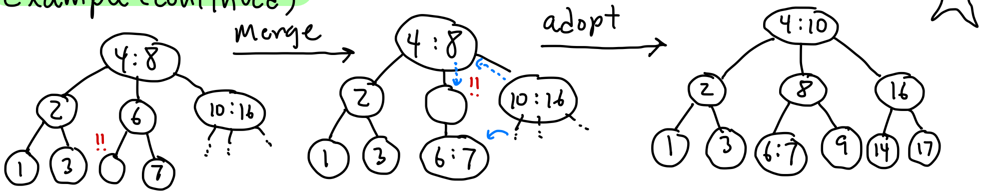


Another delete example:

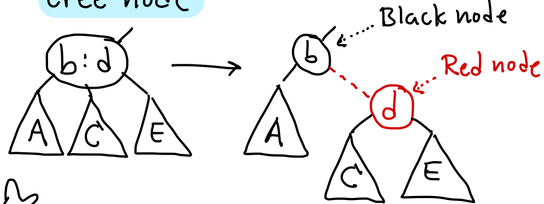


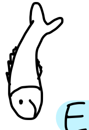


Example (continued)



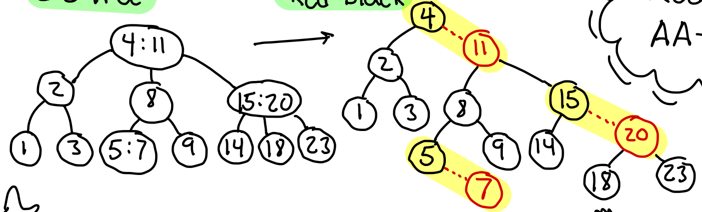
Encoding 3-node as binary tree node





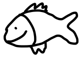
Example:

2-3 Tree: **Red-Black:**



Rules:

① Every node labeled red/black
② Root is black
③ Nulls treated as if black
④ If node is red, both children are black
⑤ Every path from root to null has same no. of black



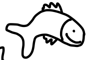
Some history:

2-3 Trees: Bayer 1972

Red-black Trees: Guibas & Sedgewick 1978 (a binary variant of 2-3)

Rumor - Guibas had two pens - red & black to draw with

Red-Black and AA-Trees I



Lemma: A red-black tree with n keys has height O(log n)

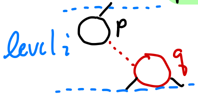
Proof: It's at most twice that of a 2-3 tree.

Q: Is every Red-Black Tree the encoding of some 2-3 tree?

AA-Trees: Simpler to code

- **No null pointers**: Create a **sentinel node**, nil, and all nulls point to it → **nil**:

- **No colors**: Each node stores **level number**. Red child is at same level as parent.



q is red ⟺ q.level == p.level

What we need are stricter rules!

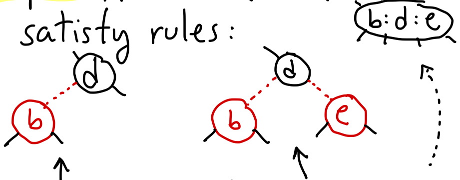
AA-tree:

Arne Anderson 1993

New rule:

⑥ Each red node can arise only as right child (of a black node)

Nope! Alternatives that satisfy rules:



A "**left-skewed**" encoding

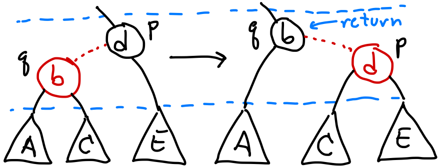
Corresponds to **2-3-4 trees**

Red-Black + AA Trees II

Restructuring Ops:

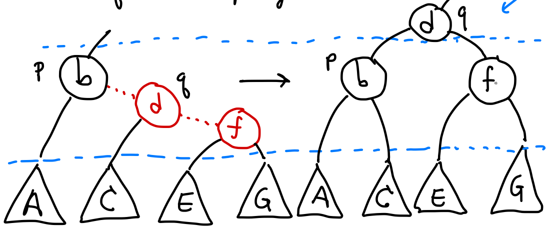
Skew: Restore right skew

→ If black node has red left child, rotate



How to test? `p.left.level == p.level`

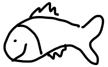
Split: If a black node has a right-right red chain, do a left rotation on it right child q, and move q's level up by one.



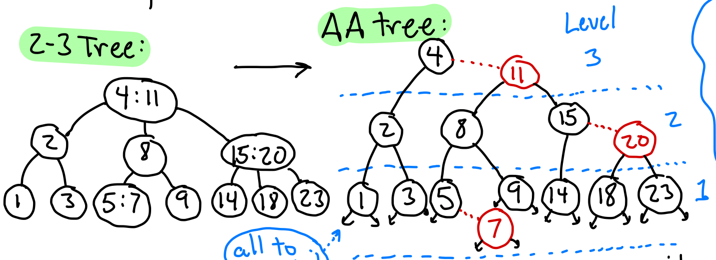
what 2-3 op does this remind you of? :o

How to test?

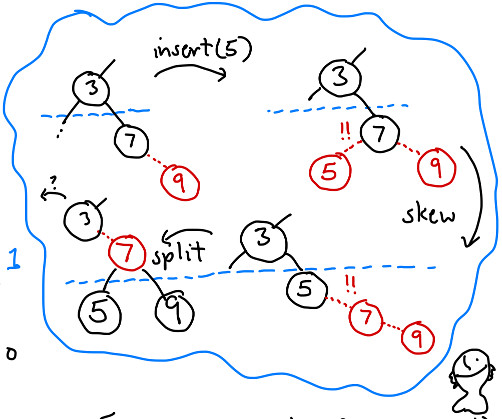
$p.level == \underbrace{p.right.level ==}_{\text{not needed (levels are monotone)}} p.right.right.level$



Example:

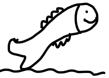


all to nil



AA Insertion:

- Find the leaf (as usual)
- Create new red node
- Back out applying skew + split

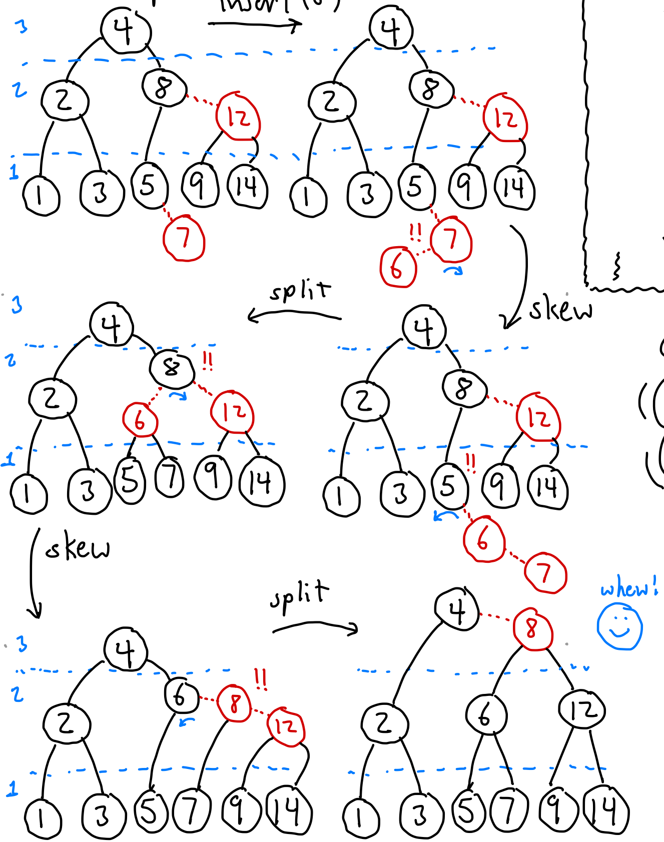


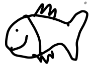
```
AANode skew(AANode p) {
    if (p == nil) return p
    if (p.left.level == p.level) {      right rotate p
        AANode q = p.left               ←
        p.left = q.right ; q.right = p
        return q      ← new subtree root
    } else return p  ← everything's fine
}
```

```
AANode split(AANode p) {
    if (p == nil) return p
    if (p.right.right.level == p.level) {
        AANode q = p.right
        p.right = q.left   ← left rotation at q
        q.left = p
        q.level += 1  ← move q up a level
        return q
    } else return p  ← all okay
}
```

Red-Black and AA Trees III

Example:

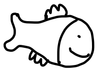




```
AANode insert(Key x, Value v, AANode p) {
    if (p == nil)
        p = new AANode(x, v, 1, nil, nil)
    else if (x < p.key) ... insert on left
    else if (x > p.key) ... insert on right
    else Duplicate Key!
    return split(skew(p))
}
```

leaf level

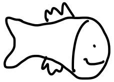
left, right



Deletion:

Two more helpers:

updateLevel: If p's level exceeds $\ell = 1 + \min(p.left.level,\ p.right.level)$ then set p's level to ℓ & also p's right child

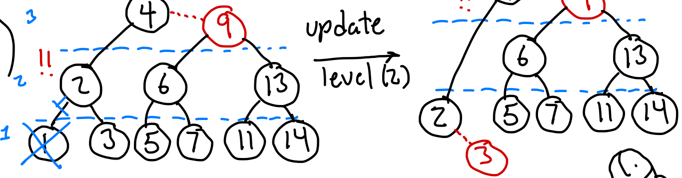


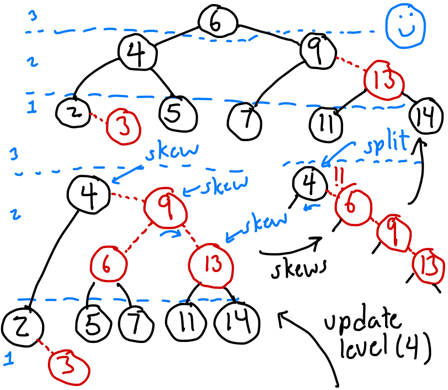
fixAfterDelete(p):

- update p's level
- skew(p), skew(p.right), skew(p.right.right)
- split(p), split(p.right)

deletion: Same as AVL deletion, but end with:
return fixAfterDelete(p)

Example: delete(1)





Treaps I

Randomized Data Structures

- Use a random number generator
- Running in expectation over all random choices
- Often simpler than deterministic

History:

1989: Seidel + Aragon
[Explosion of randomized algorithms]

Later discovered this was already known: Priority Search Trees from different context (geometry) McCreight 1980

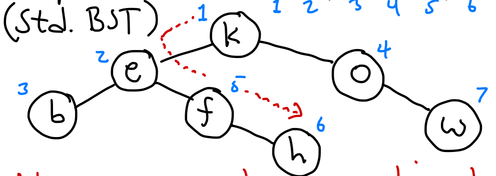
Intuition:

- Random insertion into BSTs ⇒ $O(\log n)$ expected height
- Worst case can be very bad $O(n)$ height
- Treap: A tree that behaves as if keys are inserted in random order

Example: Insert: k, e, b, o, f, h, w

1 2 3 4 5 6 7

(Std. BST)



Along any path - Insertion times increase

Obs: In a standard BST, keys are by inorder + insert times are in heap order (parent < child)

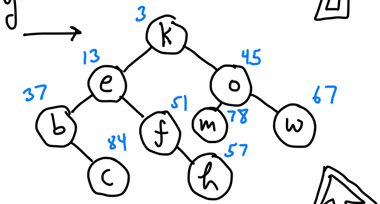
Treap: Each node stores a key + a random priority.
Keys are in inorder.
Priorities are in heap order

? Is it always possible to do both?

Yes: Just consider the corresponding BST

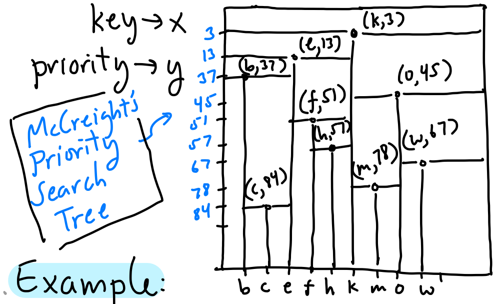
Example:

Key	Priority
b	37
c	84
e	13
f	51
h	57
k	3
m	78
o	45
w	67



Geometric Interpretation:

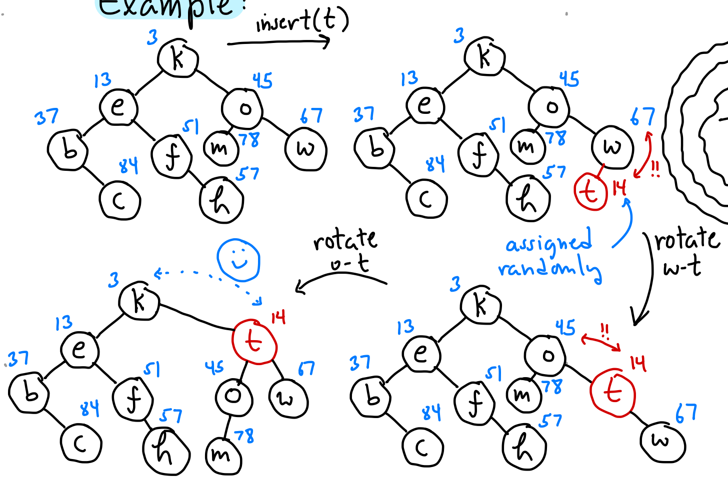
key → x
priority → y



Treaps II

Insertion: As usual, find the leaf + create a new leaf node.
- Assign random priority
- On backing out - check heap order + rotate to fix.

Example:



Deletion: (Cute solution) Find node to delete. Set its priority to +∞. Rotate it down to leaf level + unlink.

Theorem: A treap containing n entries has height O(log n) in expectation (averaged over all assignments of random priorities)

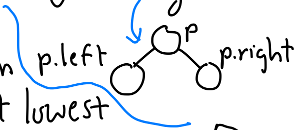
Proof: Follows directly from BST analysis

Implementation: (See pdf notes)

Node: Stores priority + usual...

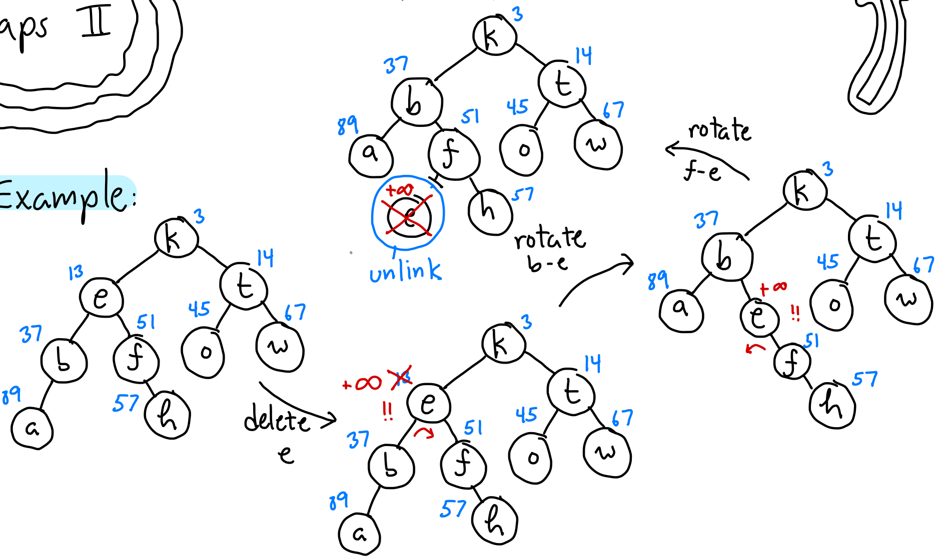
Helpers:

lowest priority(p) returns node of lowest priority among:



restructure: performs rotation (if needed) to put lowest priority node at p.

Example:



Skip Lists I

Sorted linked lists:

- Easy to code
- Easy to insert/delete
- Slow to search... $O(n)$

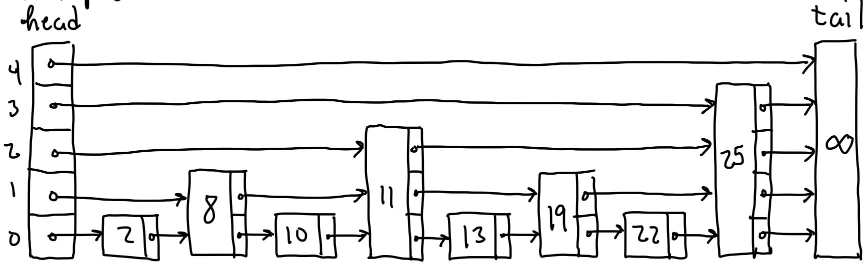
Idea: Add extra links to skip

How to generalize?

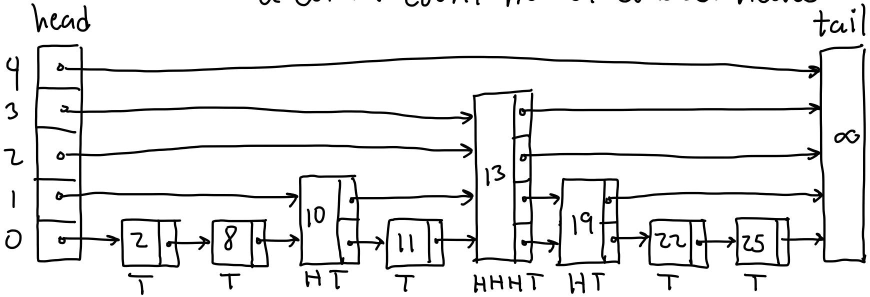
Ideal Skip List:

- Organize list in levels
- Level 0: Everything
  - 1: Every other
  - 2: Every fourth
  - ⋮
  - i: Every 2^i

Example:



Too rigid → Randomize! To determine level - toss a coin + count no. of consec. heads:



Node Structure: (Variable sized)

```
class SkipNode {
    Key key
    Value value
    SkipNode[] next
}
```

In constructor, set level and size

```
Value find(Key x) {
    i = topmost Level                          current node
    SkipNode p = head
    while (i ≥ 0) {                            until we hit base level
        if (p.next[i].key ≤ x) p = p.next[i]   advance horizontal
        else i--                               ← drop down a level
    }                                          ← we are at base level
    if (p.key == x) return p.value
        else return null
}
```

Skip Lists II

Thm: A skip list with n nodes has $O(\log n)$ levels in expectation

Proof: Will show that probability of exceeding $c\cdot\lg n$ is $\le 1/n^{c-1}$.

→ Prob that any given node's level exceeds ℓ is $1/2^\ell$ [ℓ consecutive heads]

→ Prob that any of n node's level exceeds ℓ is $\le n/2^\ell$ [n trials with prob $1/2^\ell$]

→ Let $\ell = c\cdot\lg n$ ($\lg \equiv \log_2$) Prob that max level exceeds $c\cdot\lg n$ is:

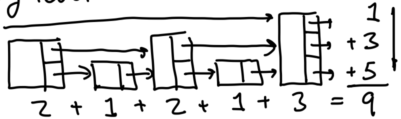
$$\le n/2^\ell = n/2^{(c\cdot\lg n)}$$
$$= n/(2^{\lg n})^c$$
$$= n/n^c = 1/n^{c-1}$$

□

Obs: Prob. level exceeds $3\cdot\lg n$ is $\le 1/n^2$.
(If $n \ge 1{,}000$, chances are less than 1 in million!)

Thm: Total space for n-node skip list is $O(n)$ expected.

Proof: Rather than count node by node, we count level by level:



- Let n_i = no. of nodes that contrib. to level i.
- Prob that node at level $\ge i$ is $1/2^i$
- Expected no. of nodes that contrib. to level $i = n/2^i$
  $\Rightarrow E(n_i) = n/2^i$

Total space (expected) is:

$$E\left(\sum_{i=0}^{\infty} n_i\right) = \sum_{i=0}^{\infty} E(n_i) = \sum_{i=0}^{\infty} n/2^i$$
$$= n\sum_{i=0}^{\infty} 1/2^i = 2n$$

□

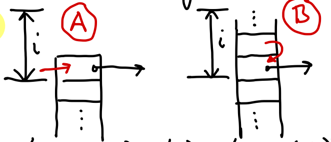
Thm: Expected search time is $O(\log n)$

Proof:
- We have seen no. levels is $O(\log n)$
- Will show that we visit 2 nodes per level on average

Obs – Whenever search arrives first time to a node, its at top level. (Can you see why?)

Def: $E(i)$ = Expect. num. nodes visited among top i levels.

Cases:



$$E(i) = \underset{\uparrow\ \text{current node}}{1} + (\text{Prob}(Ⓐ))\underset{\uparrow\ \text{same level}}{E(i)} + (\text{Prob}(Ⓑ))\underset{\uparrow\ \text{from prior level}}{E(i-1)}$$
$$= 1 + \tfrac{1}{2}E(i) + \tfrac{1}{2}E(i-1)$$
$$\Rightarrow E(i)(1-\tfrac{1}{2}) = 1 + \tfrac{1}{2}E(i-1)$$
$$\Rightarrow E(i) = [1 + \tfrac{1}{2}E(i-1)]2 = 2 + E(i-1)$$

Basis: $E(0) = 0 \Rightarrow E(i) = 2\cdot i$

Let ℓ = max level. Total visited $= E(\ell) = 2\cdot\ell$

⇒ We visit 2 nodes per level on average. □

Skip Lists III

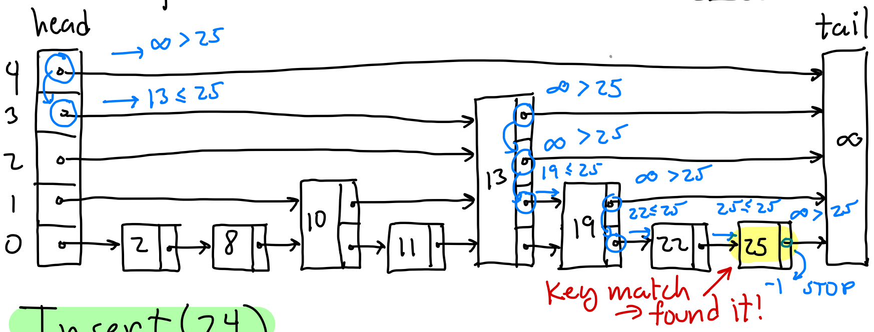
Delete:

- Start at top
- Search each level saving last node < key
- On reaching node at level 0, remove it and unlink from saved pointers

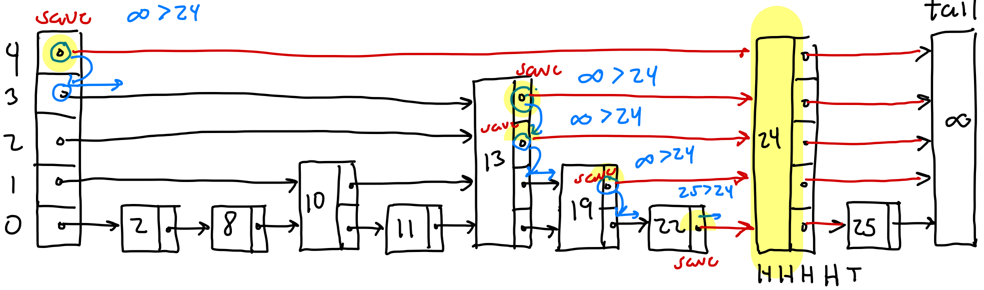
Insert: (Similar to linked lists)

- Start at top level
- At each level:
  - Advance to last node ≤ key
  - Save node + drop level
- At level 0:
  - Create new node (flip coins to determine height)
  - Link into each saved node

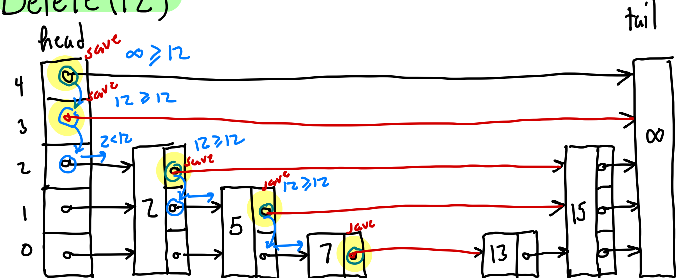
Example: find(25)



Insert(24)



Delete(12)



Analysis: All operations run in time ~ find ⇒ O(log n) expected

Note: Variation in running times due to randomness only - not sequence

⇒ User cannot force poor performance.

SPLAY TREES I

Recap: Lots of search trees
- Unbalanced BSTs
- AVL Trees
- 2-3, Red-black, AA Trees
- Treaps + Skip lists

→ **Focus:** Worst-case or randomized expected case

Other/Better Criteria?

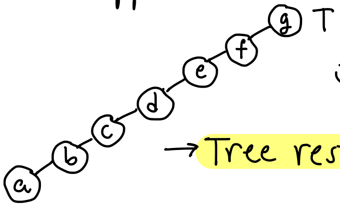
Expected case: Some keys more popular than others

Self-adjusting: Tree adapts as popularity changes

How to design/analyze?

Splay Tree: A self-adjusting binary search tree
- **No rules!** (yay anarchy!)
  - No balance factors
  - No limits on tree height
  - No colors/levels/priorities
- **Amortized efficiency:**
  - Any single op - slow
  - Long series - efficient on avg.

Intuition: Let T be an unbalanced BST + suppose we access its deepest key

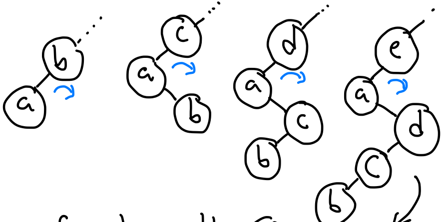


find("a")

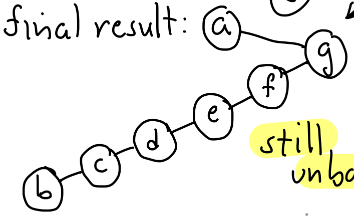
ugh!

→ **Tree restructures itself**

Idea I: Rotate "a" to top (Future accesses to "a" fast)

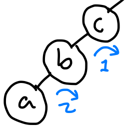


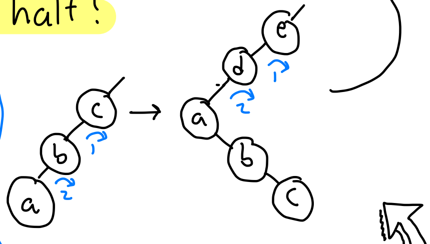
.... final result:

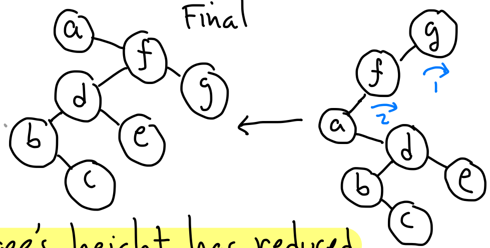


still unbalanced!

Idea II: Rotate 2 at a time - upper + lower







Tree's height has reduced by ~ half!

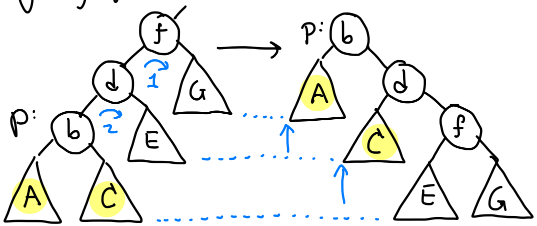
Lesson: Different combinations of rotations can:
- bring given node to root
- significantly change (improve) tree structure.

Splay Trees II

Splay(Key x):

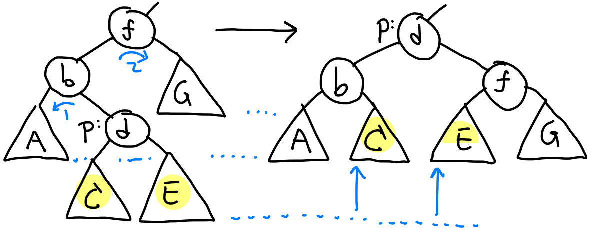
```
Node p ← find(x) [nearest node]
while (p ≠ root) {
    if (p == child of root) zig(p)
    else /* p has grand parent */
        if (p is LL or RR grand child) zigzig(p)
        else /* p is LR or RL gr. child */ zigzag(p)
```

ZigZig(p): [LL case]



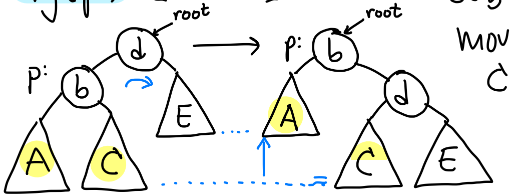
Subtrees A, C move up↑

ZIG ZAG(p): [LR case]



Subtrees C, E of p move up↑

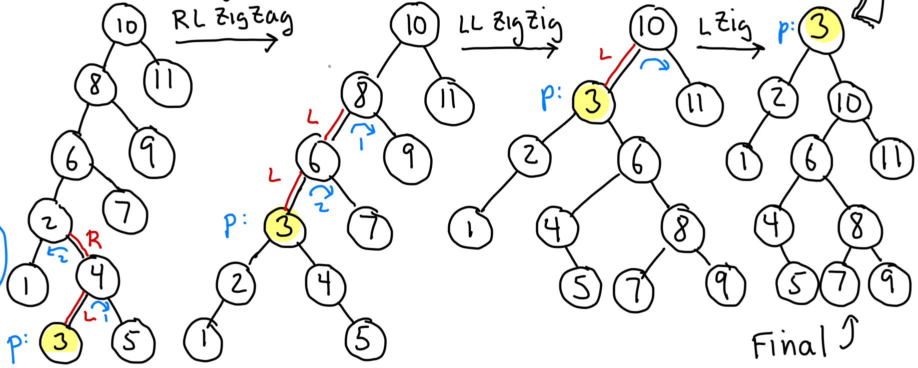
Zig(p): [L case]



Subtree A moves up↑

C unchanged

Example: splay(3)



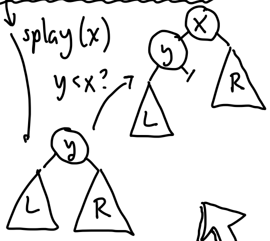
find(x):

```
splay(x)
if (root.key == x)
    found!
else not found
```

insert(x):

```
splay(x)
q = new Node(x)
if (root.key < x)
    x.left = root
    x.right = root.right
    root.right = null
else ... symmetrical...
```

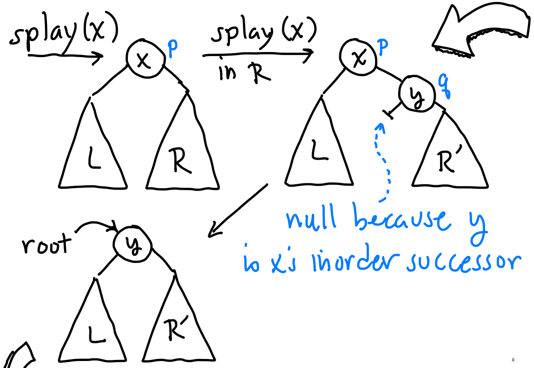
root.key ≠ x or error!



SPLAY TREES III

delete(x):

```
splay(x)  [x now at root]
p = root
if ($p.key \neq x$) error!
splay(x) in p's right subtree
q = p.right  [q's key is x's successor]
             [q.left == null]
q.left = p.left
root = q
```



Analysis:

- Amortized analysis
- Any one op might take $O(n)$
- Over a long sequence, average time is $O(\log n)$ each
- Amortized analysis is based on a sophisticated potential argument
- Potential: A function of the tree's structure
  - Balanced ⇒ Low potential
  - Unbalanced ⇒ High potential
- Every operation tends to reduce the potential

Splay Trees are Amazingly Adaptive!

Balance Theorem: Starting with an empty dictionary, any sequence of m accesses takes total time

$$O(m \log n + n \log n)$$

where n = max. entries at any time.

Static Optimality:

- Suppose key x_i is accessed with prob p_i ($\sum_{i=1}^{n} p_i = 1$)
- Information Theory: Best possible binary search tree answers queries in expected time $O(H)$ where $H = \sum_i p_i \lg 1/p_i$ ← Entropy

Static Optimality Theorem: Given a seq. of m ops. on splay tree with keys $x_1 \ldots x_n$, where x_i is accessed q_i times. Let $p_i = q_i/m$. Then total time is

$$O(m \sum_i p_i \lg 1/p_i)$$

Dynamic Finger Theorem:

Keys: $x_1 < \ldots < x_n$. We perform accesses $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$.

Let $\Delta_j = i_j - i_{j-1}$: distance between consecutive items

Thm: Total access time is

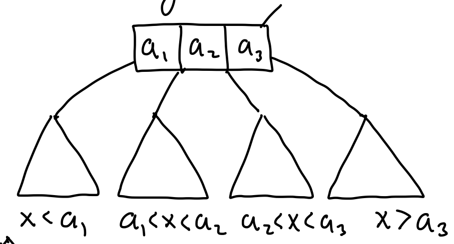
$$O(m + n \log n + \sum_{j=1}^{m} (1 + \lg \Delta_j))$$

B-Trees I

Secondary Memory:

- Most large data structures reside on disk storage
- Organized in blocks - pages
- Latency: High start-up time
- Want to minimize no. of blocks accessed

Multiway Search Trees:



B-Tree:

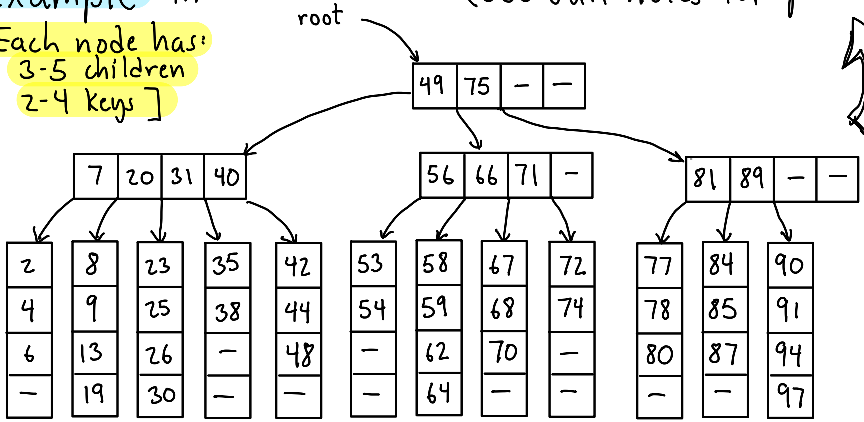
- Perhaps the most widely used search tree
- 1970 - Bayer + McCreight
- Databases
- Numerous variants

B-Tree: of order m (≥3)

- Root is leaf or has ≥ 2 children
- Non-root nodes have $\lceil m/2 \rceil$ to m children [null for leaves]
- k children ⇒ k-1 key-values
- All leaves at same level

Example: m=5

[Each node has:
3-5 children
2-4 keys]



Theorem:

A B-tree of order m with n keys has height at most $(\lg n)/\gamma$, where $\gamma = \lg(m/2)$

(See full notes for proof)

Node Structure:

constant int M=...

```
class BTreeNode {
    int       nChild      // no. of children
    BTreeNode child[M]    // children
    Key       key[M-1]    // keys
    Value     value[M-1]  // values
}
```

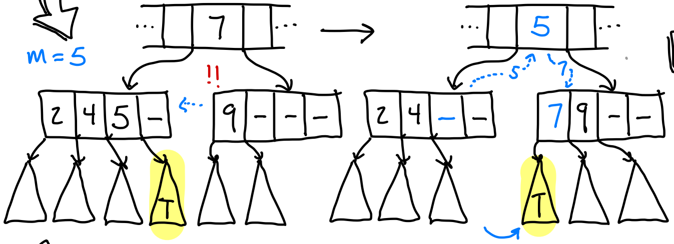
B-Trees II

B-Tree restructuring:

- Generalizes 2-3 restructure
- Key rotation (Adoption)
- Splitting (insertion)
- Merging (deletion)

Key Rotation (Adoption)

- A node has too few children $\lceil m/2 \rceil - 1$
- Does either immediate sibling have extra? $\geq \lceil m/2 \rceil + 1$
- Adopt child from sibling & rotate keys
- When applicable - preferred



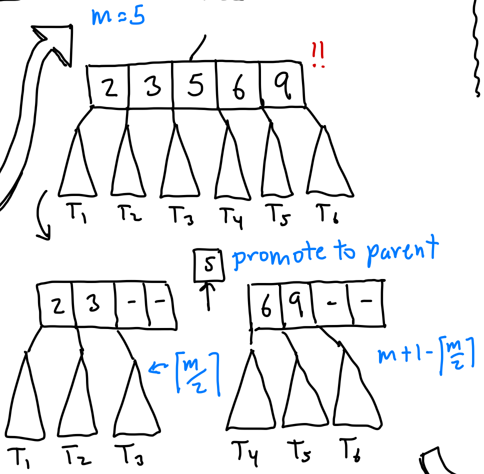
Node Splitting:

- After insertion, a node has too many children... m+1
- We split into two nodes of sizes $m' = \lceil m/2 \rceil$ and $m'' = m+1-\lceil m/2 \rceil$

Lemma: For all $m \geq 2$,

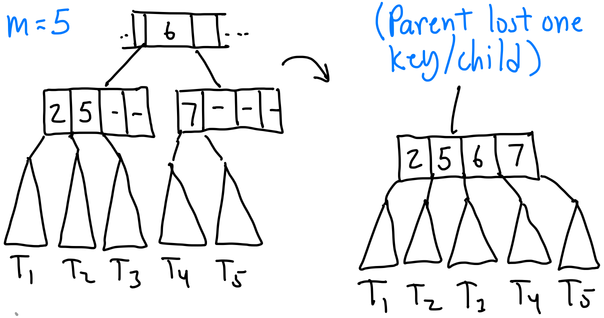
$$\lceil m/2 \rceil \leq m+1-\lceil m/2 \rceil \leq m$$

$\Rightarrow$ m' + m'' are valid node sizes



Node Merging:

- A node has too few children $\lceil m/2 \rceil - 1$
- Neither sibling has extra (both $\lceil m/2 \rceil$)
- Merge with either sibling to produce node with $(\lceil m/2 \rceil - 1) + \lceil m/2 \rceil$ child



Lemma: For all $m \geq 2$,

$$\lceil m/2 \rceil \leq 2\lceil m/2 \rceil - 1 \leq m$$

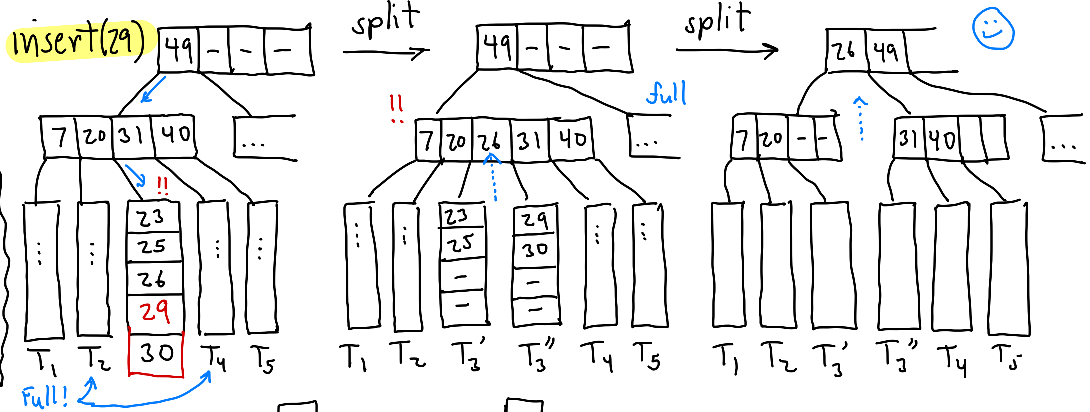
$\Rightarrow$ Resulting node is valid

B-Trees III

Insertion:

- Find insertion point (leaf level)
- Add key/value here
- If node overfull (m keys, m+1 children)
  - → Can either sibling take a child (< m)?
  - ⇒ Key rotation [done]
  - → Else, split
    - → Promotes key
    - → If root splits add new root

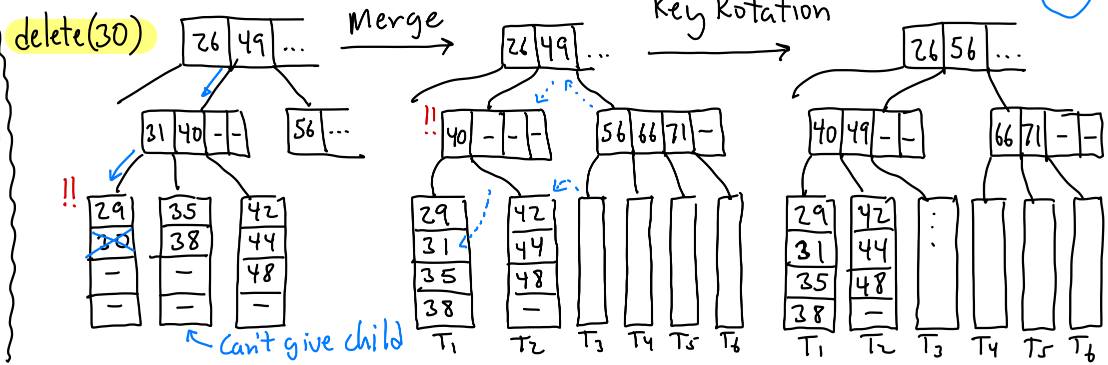
Example: m=5



Deletion:

- Find key to delete
- Find replacement/copy
- If underfull ($\lceil m/2 \rceil - 1$) child
  - → If sibling can give child
    - → Key rotation
  - → Else (sibling has $\lceil m/2 \rceil$)
    - → Merge with sibling
  - → Propagates → If root has 1 child → collapse root

Example: m=5



Scapegoat Trees:

- Arne Anderson (1989)
- Galperin + Rivest (1993) rediscovered/extended
- **Amortized analysis**
  - $O(\log n)$ for dictionary ops amortized (guaranteed for find)
  - Just let things happen
  - If subtree unbalanced – rebuild it

Overview:

Insert:

- same as standard BST
- if depth too high
  - trace search path back
  - find unbalanced node – **scapegoat**
  - rebuild this subtree

Find: Same as std BST

- Tree height $\leq \log_{3/2} n \approx 1.7 \lg n$

Recap:

- Seen many search trees
- Restructure via **rotation**
- Today: Restructure via **rebuilding**
- Sometimes rotation not possible
- Better mem. usage

Scapegoat Trees I

Delete:

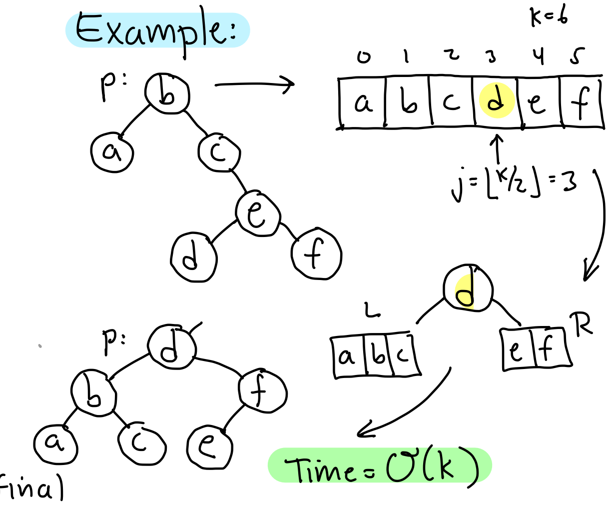
- Same as std. BST
- If num. of deletes is large rel. to n – rebuild entire tree!

How? Maintain $n, m \leftarrow 0$

Insert: n++, m++

Delete: n-- ⋯> If $m > 2n$ rebuild

Example:



Time = $O(k)$


How to rebuild?

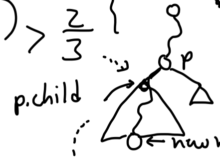
```
rebuild(p):
    - inorder traverse p's subtree -> array A[]
    - buildSubtree(A)

buildSubtree(A[0..k-1]):
    - if k=0 return null
    - j <- ⌊k/2⌋ ; x <- A[j]  median
    - L <- buildSubtree(A[0..j-1])
    - R <- buildSubtree(A[j+1..k-1])
    - return Node(x, L, R)
```

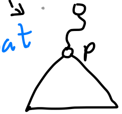
Scapegoat Trees II

Insert:

- n++; m++
- same as std BST but keep track of inserted node's depth → d
- if ($d > \log_{3/2} m$) {
  - /* rebuild event */
  - trace path back to root
  - for each node p visited, size(p) = no. of nodes in p's subtree
  - if $\frac{\text{size(p.child)}}{\text{size(p)}} > \frac{2}{3}$

    p ← rebuild(p)
  - break

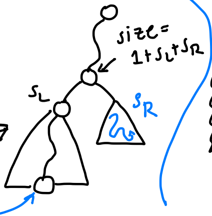




How to compute size(p)?

- Can compute it on the fly
- While backing out, traverse "other sibling"
- Too slow? No!

  → Charge to rebuild.



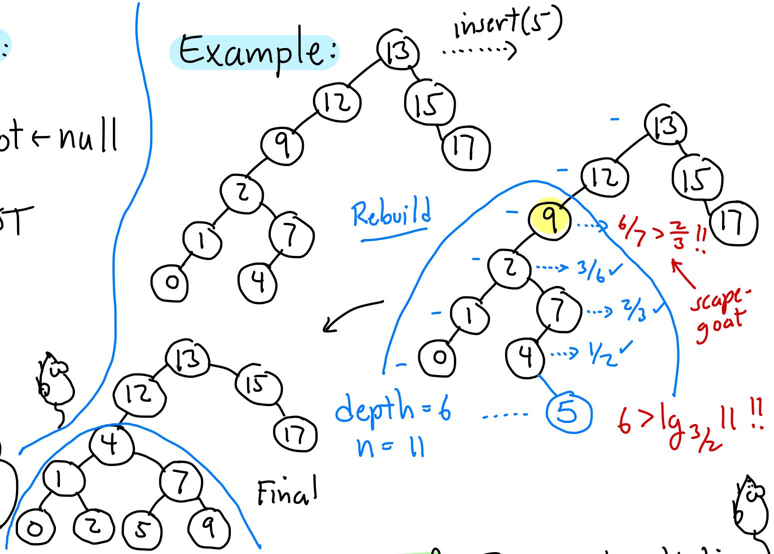
Details of Operations:

Init: n ← m ← 0 root ← null

Delete:
- Same as std BST
- n--
- if m > 2n, rebuild(root)

Time: O(n)

Example:

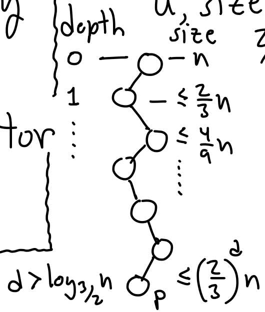


Must there be a scapegoat? yes!

Lemma: Given a binary tree with n nodes, if ∃ node p of depth $> \log_{3/2} n$, then ∃ ancestor of p that satisfies scapegoat condition

Proof: By contradiction

- Suppose p's depth $> \log_{3/2} n$ but ∀ ancestors u, $\text{size(u.child)} \le \frac{2}{3} \cdot \text{size}(u)$



$\Rightarrow$ Since p has 1 node:

$1 \le \text{size}(p) \le \left(\frac{2}{3}\right)^d n$

$\Rightarrow \left(3/2\right)^d \le n$

$\Rightarrow d \le \log_{3/2} n$ □

Scapegoat Trees

III

Theorem: Starting with an empty tree, any sequence of m dictionary operations on a scapegoat tree take time $O(m \log m)$ [Amortized: $O(\log m)$]

Proof: (Sketch)

Find: $O(\log n)$ guaranteed [Height $= O(\log n)$]

Delete: In order to induce a rebuild, number of deletes ~ number of nodes in tree

→ Amortize rebuild time against delete ops

Insert: Based on potential argument

→ It takes ~ k ops to cause a subtree to size k to be unbalanced.

→ Charge rebuild time to these operations

Weight-Balanced Trees I

Overview:

- Splay trees - **Static Optimality**
- More frequently accessed keys closer to root

$\Rightarrow$ **Weight-balanced trees**

Weight Balance:

- Given a set of **keys** $X = \{x_0, \ldots, x_{n-1}\}$
- and **values** $V = \{v_0, \ldots, v_{n-1}\}$
- and **weights** $W = \{w_0, \ldots, w_{n-1}\}$
- Assume:

$x_0 < x_1 < \ldots < x_{n-1}$ sorted

$w_i > 0$ positivity

Pseudo-Probability:

- Let: $\overline{W} = \sum_{i=0}^{n-1} w_i$ **total weight**
- Let: $p_i = w_i / \overline{W}$ **pseudo-prob**
- Obs: $\left.\begin{array}{l} 0 < p_i \leq 1 \\ \sum_i p_i = 1 \end{array}\right\} \Rightarrow$ discrete prob. distribution

Shannon's Theorem:

If p_i is the prob. of accessing x_i, any BST has expected search at least $\sum_i p_i \lg 1/p_i$ ← (Called the **entropy** of distrib)

How to (Nearly) Achieve Shannon's bound

→ Weight-balanced tree

→ For each node p:

$wt(p)$ = total weight of keys in p's subtree

$$\textbf{balance}(p) = \frac{\max(wt(p.left), wt(p.right))}{wt(p)}$$

Given $1/2 \leq \alpha \leq 1$, a BST is **α-balanced** if for all internal nodes p, $balance(p) \leq \alpha$

$\alpha = 1/2$: Perfectly balanced

$= 1$: Arbitrarily bad

$\alpha = 2/3$: A reasonable compromise

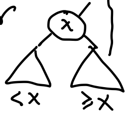
Implementation: (as extended BST)

Internal node: Stores:

Key key → splitter

float wt → total weight of entries in subtree

left, right



External Node:

Key key, ← x_i

Value value

float wt ← w_i

Weight-Balanced Trees II

How to maintain balance?

Options:

- **Rotations**: Similar to AVL trees (single + double) ····> BB[α] trees
- **Rebuild subtrees**: Similar to scapegoat

Balance by Rebuilding:

Given an array $A[0..k-1]$ of external nodes:

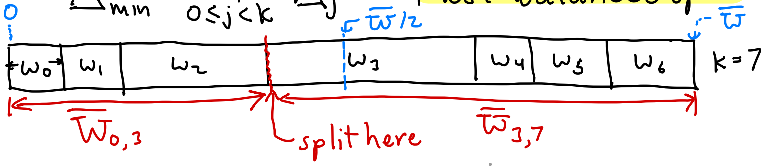
$A[i].key$, $A[i].value$, $A[i].wt$

- Assume keys are sorted
- Assume weights > 0

Weight-based median:

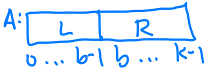
- Select splitter to minimize left-right weight difference
- Let $\overline{W} = \sum_{i=0}^{k-1} A[i].wt$ ← Total wt
- Let $\overline{W}_{i,j} = \sum_{m=i}^{j-1} A[i].wt$ ← Total wt of $A[i..j-1]$ (←···· note!)
- Let $\Delta_j = |\overline{W}_{0,j} - \overline{W}_{j,k}|$ ← Absolute diff if we split $A[0..j-1]$ & $A[j..k-1]$
- **Goal**: Split at $0 \le j < k$ that minimizes:

$$\Delta_{min} = \min_{0 \le j < k} \Delta_j \quad \leftarrow \text{Most balanced split}$$

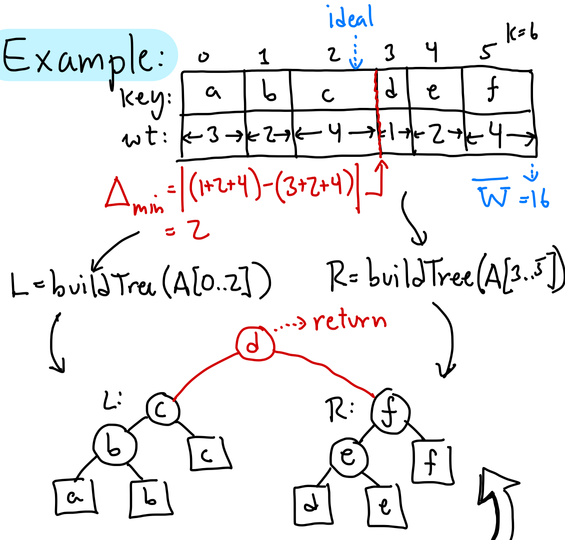


buildTree (A[0..k-1])

```
if (k == 1) return A[0]   /* base case */
$\overline{W} = \sum_{i=0}^{k-1} A[i].wt$   /* total weight */
Init: b = 0; Lwt = 0; Rwt = $\overline{W}$; $\Delta_{min} = \overline{W}$
for (i = 0 ... k-1)
    Lwt += A[i].wt ; Rwt -= A[i].wt
    $\Delta$ = |Rwt - Lwt|   /* weight difference */
    if ($\Delta < \Delta_{min}$) { b = i+1;  $\Delta_{min} = \Delta$ }
L = buildTree(A[0..b-1])
R = buildTree(A[b..k-1])
return new IntNode(A[b].key, L, R)
```



Example:



Weight-Balanced Trees III

But it is pretty close! ☺

Theorem: (Mehlhorn '77) The above balanced split algorithm produces a tree whose exp. search time is

$$\leq H + 3$$

where H = entropy bound.

Dictionary Operations:

→ Balance by destroying + rebuilding - **Jackhammer Trees**

Find: Same as usual. Tree height $\leq \log_{3/2} n$, so $O(\log n)$ time - guaranteed.

Insert/Delete: Start same as standard BST

→ After operation completes **check + rebuild**

Analysis:

Does this algorithm produce the **optimal tree** (w.r.t. expected case search time)?

- No. ☹ The optimal BST can be computed by **dynamic programming**

CMSC 451

Check + Rebuild:

- When returning from recursive calls, update each node's weight

$$p.wt \leftarrow p.left.wt + p.right.wt$$

- Starting at root, walk down search path. Stop at first node p s.t.

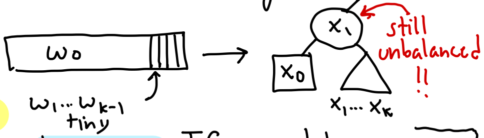
$$\text{balance}(p) > \alpha$$

Recall def earlier

Given by designer e.g. $\alpha = 2/3$

Bad weight distributions?

- If a weight is very large relative to neighbors, rebalance may be ineffective



Lemma: If weights are "nice" (not too much variation), insert + delete run in $O(\log n)$ amortized time.

If no such p found - Great! Tree is balanced

Else: **Jackhammer!**

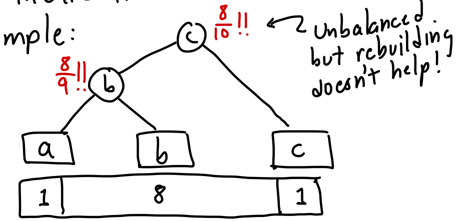
- Traverse p's subtree inorder, store extern nodes in array A[0..k-1]
- Replace p's subtree with

buildTree(A)

Weight-Balanced Trees IV

Very heavy entries:

- If an entry's weight is too high, rebuilding is ineffective
- Example:



- This tree is best possible!
- Exemption: Don't rebuild if a key's weight is very high

For node p: max(p) = max weight in p's subtree

$$\text{max-ratio}(p) = \frac{\max(p)}{\text{weight}(p)}$$

Given parameter $0 < \beta < 1$, a node is β-exempt if $\text{max-ratio}(p) > \beta$

Dictionary Operations:

- find: as usual
- insert: insert as usual but rebuild if needed
- delete: delete as usual but rebuild if needed

(α,β)-balance: Every internal node p is either α-balanced or β-exempt

Lemma: For any set of weighted entries, ∃ an (α,β)-balanced BSTree if $\frac{1}{2} < \alpha < 1$ and $\beta < 2\alpha - 1$

E.g. $\alpha = 2/3$, $\beta = 1/4$

When to rebuild?

- When "backing out" from insert/delete, update node weights
- Walk down search path from root [opposite from scapegoat!]
- If any node p is out of balance:

$\text{balance}(p) > \alpha$

-and-

$\text{max-ratio}(p) \leq \beta$

then:

- Rebuild p:
  - Traverse p's subtree inorder
  - Collect external nodes in array A[0..k-1]
  - replace p with buildTree(A)

Hashing I

Recap: So far, **ordered dicts.**

- insert, delete, find
- **Comparison-based**: <, ==, >
- getMin, getMax, getK, findUp...
- Query/Update time: $O(\log n)$
  - → Worst-case, amortized, random.

→ Can we do better? $O(1)$?

Hashing: (Unordered) dictionary

- stores key-value pairs in **array** **table[0..m-1]**
- supports basic dict. ops. (insert, delete, find) in **$O(1)$ expected time**
- does not support ordered ops (getMin, findUp, ...)
- simple, practical, widely used

Overview:

- To store n keys, our table should (ideally) be a bit larger (e.g., $m \geq c \cdot n$, $c = 1.25$)
- **Load factor:**
  $$\lambda = n/m$$
- Running times increase as $\lambda \to 1$
- **Hash function:**
  $$h : Keys \longrightarrow [0..m-1]$$
  - → Should **scatter** keys random.
  - → Need to handle **collisions** → $x \neq y$ but $h(x) = h(y)$

Good Hash Function:

- Efficient to compute
- Produce few collisions
  - Use every bit in key
  - Break up natural clusters

E.g. Java variable names: temp1, temp2, temp3

table:

Common Examples:

Assume keys can be interpreted as ints

- **Division hash:**
  $$h(x) = x \bmod m$$
- **Multiplicative hash:**
  $$h(x) = (ax \bmod p) \bmod m$$
  a, p - large prime numbers
- **Linear hash:**
  $$h(x) = ((ax + b) \bmod p) \bmod m$$
  a, b, p - large primes

Why "mod p mod m"?

- modding by a large prime scatters keys
- m may not be prime (e.g. power of 2)

Universal Hashing:

Even better → randomize!

- Let H be a **family** of hash fns
- Select $h \in H$ randomly
- If $x \neq y$ then $Prob(h(x) = h(y)) = 1/m$

E.g. Let p - large prime, $a \in [1..p-1]$, $b \in [0..p-1]$ **all random**

- $h_{a,b}(x) = ((ax + b) \bmod p) \bmod m$

Hashing II

Overview:

- Separate Chaining
- Open Addressing:
  - Linear probing
  - Quadratic probing
  - Double hashing

simple/slow ↓ ↓ complex/fast

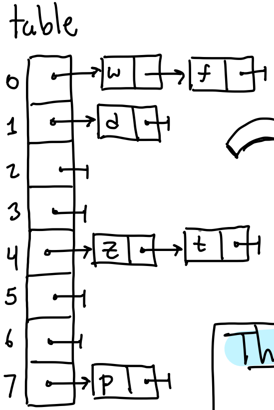
Separate Chaining:

table[i] is head of linked list of keys that hash to i.

Example:

Keys (x)	h(x)
d	1
z	4
p	7
w	0
t	4
f	0

m = 8



Analysis: Recall load factor

$\lambda = n/m$

n = # of keys

m = table size

Collision Resolution:

If there were no collisions hashing would be trivial!

insert(x,v) → table[h(x)] = v

find(x) → return table[h(x)]

delete(x) → table[h(x)] = null

S_{sc} = Expected search time if x found (successful)

U_{sc} = Expect. search time if x not found (unsuccessful)

Thm: $S_{sc} = 1 + \lambda/2$ $U_{sc} = 1 + \lambda$

Proof: On avg. each list has $n/m = \lambda$

success: 1 for head + half the list

unsuccess: 1 " " + all the list

If $\lambda < \lambda_{min}$ or $\lambda > \lambda_{max}$? Rehash!

- Alloc. new table size = n/λ_0
- Compute new hash fn h
- Copy each x,v from old to new using h
- Delete old table

Token-based – See latex notes!

Thm: Amortized time for rehashing is $1 + (2\lambda_{max}/(\lambda_{max} - \lambda_{min}))$

How to control λ?

-Rehashing: If table is too dense/too sparse, realloc. to new table of ideal size

Designer: $\lambda_{min}, \lambda_{max}$ - allowed λ values

$\lambda_0 = \frac{\lambda_{min} + \lambda_{max}}{2}$ "ideal"

If $\lambda < \lambda_{min}$ or $\lambda > \lambda_{max}$ ...

Hashing III

Collision Resolution: (cont.)

- Separate chaining is efficient, but uses extra space (nodes, pointers, ...)
- Can we just use the table itself?

→ **Open Addressing**

Open Addressing:

- Special entry ("empty") means this slot is unoccupied
- Assume $\lambda \leq 1$
- To insert key:

  check: $h(x)$ if not empty try
  $h(x)+i_1$
  $h(x)+i_2$
  $\vdots$

  $\langle i_1, i_2, i_3, \ldots \rangle$ – **Probe sequence**
- What's the best probe sequence?

Linear Probing:

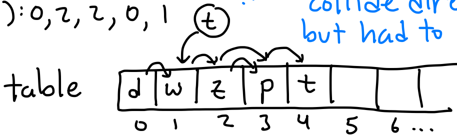
$h(x), h(x)+1, h(x)+2, \ldots$

until finding first available

Simple, but is it good?

x: d, z, p, w, t

h(x): 0, 2, 2, 0, 1



t did not collide directly but had to probe 3 times!

Analysis:

Let S_{LP} = expected time for successful search

U_{LP} = " " unsuccessful "

Thm:
$$S_{LP} = \frac{1}{2}\left(1 + \frac{1}{1-\lambda}\right)$$
$$U_{LP} = \frac{1}{2}\left(1 + \frac{1}{1-\lambda}\right)^2$$

Obs: As $\lambda \to 1$ times increase rapidly

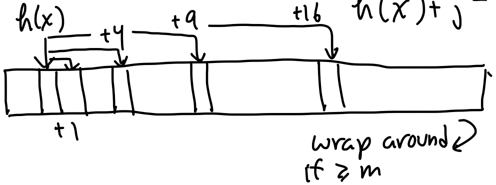
Clustering

- Clusters form when keys are hashed to nearby locations
- Spread them out!

Quadratic Probing:

$h(x), h(x)+1, h(x)+4, h(x)+9, \ldots$

$h(x)+j^2$



Analysis:

Improves secondary clustering

- May fail to find empty entry (Try m=4. $j^2 \bmod 4 = 0$ or 1 but not 2 or 3)
- How bad is it? It will succeed if $\lambda < 1/2$.

Thm: If quad. probing used + m is prime, the the first $\lfloor m/2 \rfloor$ probe locations are distinct.

Pf: See latex notes.

Hashing IV

Recap:

Separate Chaining: Fastest but uses extra space (linked list)

Open Addressing:
- Linear probing:
- Quadratic probing:

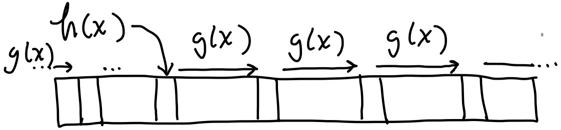
} clustering

Double Hashing:

(Best of the open-addressing methods)

- Probe sequence det'd by second hash fn. – $g(x)$

$$h(x)+\{0, g(x), 2\cdot g(x), 3\cdot g(x) \ldots\} \quad [\text{mod } m]$$



(until finding an empty slot)

Why does bust up clusters?

Even if $h(x) = h(y)$ [collision] it is very unlikely that $g(x) = g(y)$

⇒ Probe sequences are entirely different!

Analysis:

Defs:

S_{DH} = Expected search time of doub. hash. if successful

U_{DH} = Exp. if unsuccessful

Recall: Load factor $\lambda = n/m$

Thm: $S_{DH} = 1/\lambda \ln(1/(1-\lambda))$
$U_{DH} = 1/(1-\lambda)$

→ Proof is nontrivial (skip)

λ:	0.5	.075	0.95	0.99
U_{DH}:	2	4	20	100
S_{DH}:	1.39	1.89	3.15	4.65

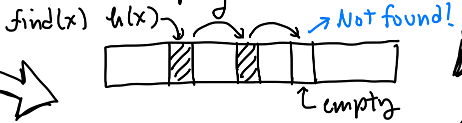
Very efficient!

Dictionary Operations:

Insert(x,v): Apply probe sequence until finding first empty slot.
- Insert (x,v) here.

(If x found along the way ⇒ duplicate key error!)

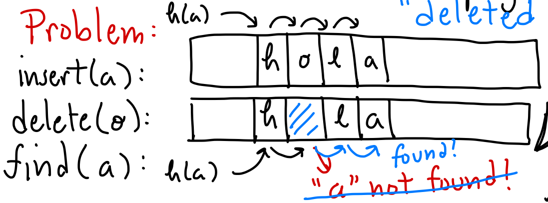
Find(x): Visit entries on probe sequence until:
- found x ⇒ return v
- hit empty ⇒ return null



Delete(x): Apply find(x)
→ Not found ⇒ error
→ Found ⇒ set to ~~"empty"~~ "deleted"

~~Is this right??~~

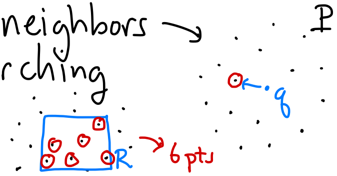
Problem:

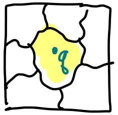


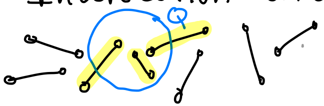
Quadtrees & kd-Trees I

Geometric Search:

- Nearest neighbors
- Range searching
- Point Location
- Intersection Search





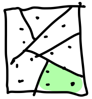


⋮

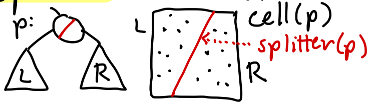
So far: 1-dimensional keys

- Multi-dimensional data
- Applications:
  - Spatial databases + maps
  - Robotics + Auton. Systems
  - Vision/Graphics/Games
  - Machine Learning
- …

Partition Trees:



- Tree structure based on hierarchical space partition
- Each node is associated w. a region – cell
- Each internal node stores a splitter – subdivides the cell



- External nodes store pts.

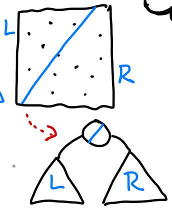
Point: A d-vector in $\mathbb{R}^d$

$p = (p_1, \dots, p_d) \quad p_i \in \mathbb{R}$

Multi-Dim vs. 1-dim Search?

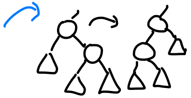
Similarities:

- Tree structure
- Balance $O(\log n)$
- Internal nodes – split
- External nodes – data



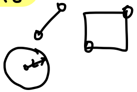
Differences:

- No (natural) total order
- Need other ways to discriminate + separate
- Tree rotation may not be meaningful



Representations:

- **Scalars:** Real numbers for coordinates, etc. float
- **Points:** $p = (p_1, \dots, p_d)$ in real d-dim space $\mathbb{R}^d$
- **Other geom objects:** Built from these

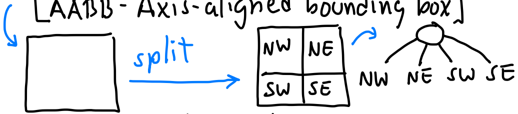


```
class Point {
   float[] coord        // coords
   Point(int d)
         ....> coord = new float[d]
   int getDim() ...> coord.length
   float get(int i) ...> coord[i]
   .... others: equality, distance
            toString ...
}
```

Quadtrees & kd-Trees II

Quadtrees: (abstractly)

- Partition trees
- **Cell**: Axis-parallel rectangle
  [AABB - "Axis-aligned bounding box]

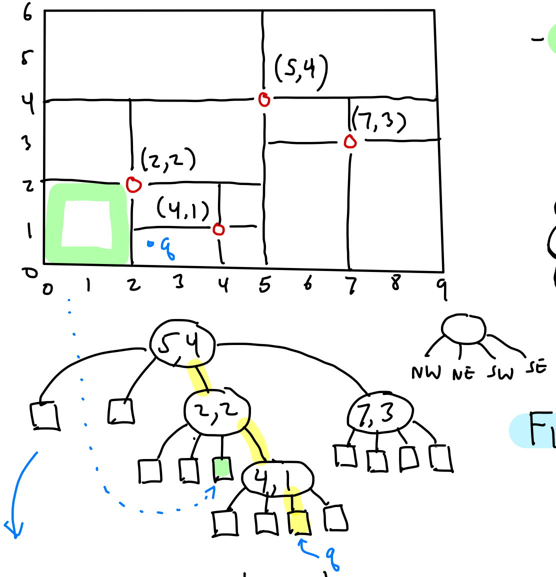


- **Splitter**: Subdivides cell into four (genlly 2^d) subcells

Point Quadtree:

- Each internal node stores a point
- Cell is split by horiz. + vertic. lines through point

(5,4)
(2,2)
(7,3)
(4,1)



Each external node corresps. to cell of final subdivision

Find/Pt Location:

Given a query point q, is it in tree, and if not which leaf cell contains it?

→ Follow path from root down (generalizing BST find)

Quadtrees - Analysis

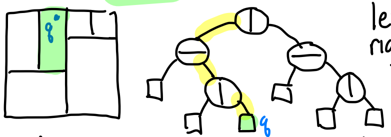
- Numerous variants! PR, PMR, QR, QX, ... see Samet's book
- Popular in 2-d apps (in 3-d, octtrees)
- Don't scale to high dim
  - out degree = 2^d
- What to do for higher dims?

History: Bentley 1975

- called it 2-d tree ($\mathbb{R}^2$)
  3-d tree ($\mathbb{R}^3$)
  ⋮
- In short **kd-tree** (any dim)
- Where/which direction to split?
  → next

kd-Tree: Binary variant of quadtree

- **splitter**: Horiz. or vertic. line in 2-d (orthogonal plane o.w.)
- **cell**: Still AABB



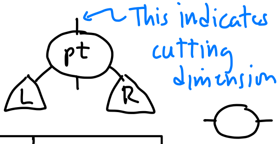
left: left/below
right: right/above

Quadtrees & kd-Trees III

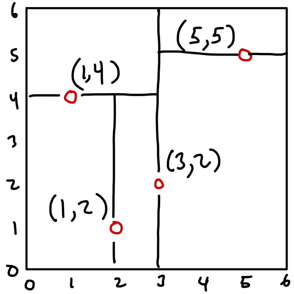
Kd-Tree Node:

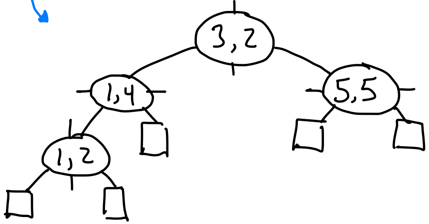
```
class KDNode {
    Point pt // splitting point
    int cutDim // cutting coordinate
    KDNode left // low side
    KDNode right // high side
}
```

Example:



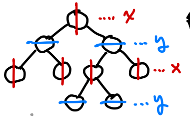
(3,2)
(1,4)
(5,5)
(1,2)



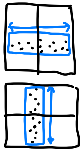


How do we choose cutting dim?

- **Standard kd-tree**: cycle through them (eg. d=3: 1,2,3,1,2,3...) based on tree depth



- **Optimized kd-tree**: (Bentley)
  - Based on widest dimension of pts in cell.



Find:

- Descend the tree
- Compare query pt with node pt along cutDim

```
class KDNode {
    :
    boolean onLeft(Point q)
    :{return q[cutDim] < pt[cutDim]}
}
```

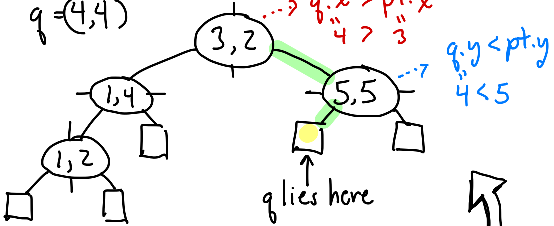
```
Value find(Point q, KDNode p) {
    if (p == null) return null;
    else if (q == p.pt)   ···> all coords match?
        return p.value
    else if (p.onLeft(q))
        return find(q, p.left)
    else
        return find(q, p.right)
}
```

Theorem: If pts are inserted in random order, expected height is O(log n)

Analysis: Find runs in time O(h), where h is height of tree.

Example: find(q) $\xrightarrow{\text{calls}}$ find(q, root)

q = (4,4)



Quadtrees & kd-Trees IV

```
KDNode insert(Point x,
    Value v, KDNode p, int cd){
  if(p==null)  //fell out?
    p = new KDNode(x,v,cd)
          // new leaf node
  else if(p.pt == x)
    Error! Duplicate key
  else if(p.onLeft(x))
    p.left=insert(x,v,p.left,(cd+1)%dim)
  else
    p.right=insert(x,v,p.right,
                   (cd+1)%dim)
  return p
}
```

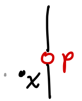
cutting dimension to use

dimension of point

Kd-Tree Insertion:

(Similar to std. BSTs)

- Descend tree until
  - → find pt → Error-duplicate
  - → falling out ← (Although we draw extended trees, lets assume standard trees)
    - → create new node
    - → set cutting dim



Deletion:

- Descend path to leaf
- If found:
  - leaf node → just remove
  - internal node
    - → find replacement
    - → copy here
    - → recur. delete replacement

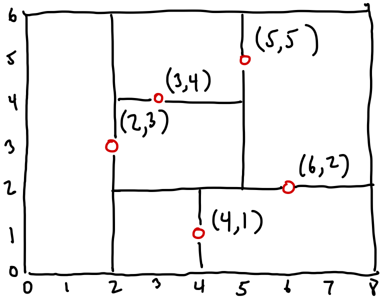
This is the hardest part. See Latex notes.

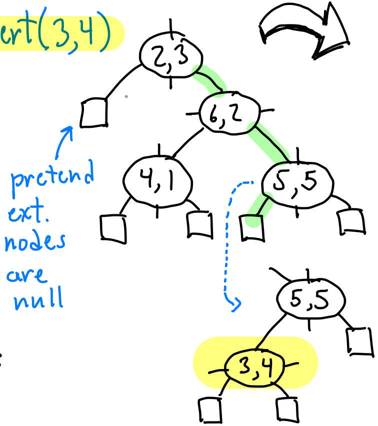
Rebalance by Rebuilding:

- Rebuild subtrees as with scapegoat trees
- O(log n) amortized
- Find: O(log n) guaranteed.

Example:

insert(3,4)





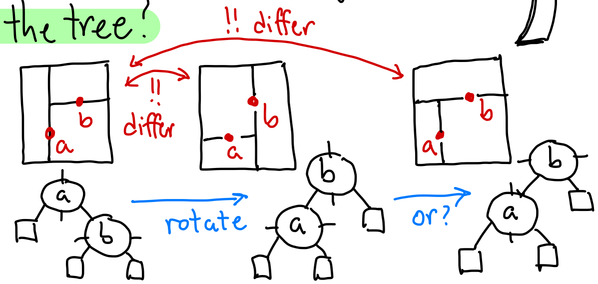
Analysis:

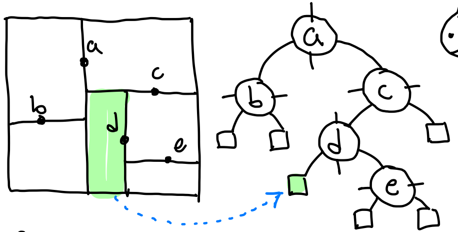
Run time: O(h)

Tree height

Can we balance the tree?

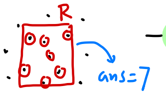
- Rotation does not make sense



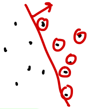


Queries?

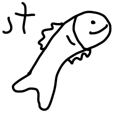
- Orthogonal range queries
  - Given query rect. (AABB) count/report pts in this rect.



- Other range queries?
  - Circular disks
  - Halfplane
  - ⋮



- Nearest neighbor queries
  - Given query pt, return closest pt in the set
  - Find k^{th} closest point
  - Find farthest point from q

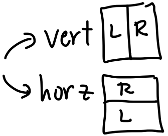


This Lecture: $O(\sqrt{n})$ time alg for orthog. range counting queries in $\mathbb{R}^2$

→ General $\mathbb{R}^d$: $O(n^{1-1/d})$

Kd-Trees:

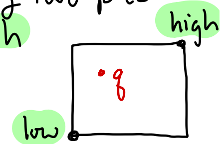
- Partition trees
- Orthogonal split
- Alternate cutting dimension x, y, x, y, ...
- Cells are axis-aligned rectangles (AABB)



Kd-Tree Queries I

Axis-Aligned Rect in $\mathbb{R}^d$

- Defined by two pts: low, high

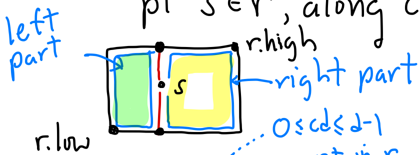


- Contains pt $q \in \mathbb{R}^d$ iff

$low_i \le q_i \le high_i$ $1 \le i \le d$

Rectangle methods for kd-cells:

- Split a cell r by a split pt $s \in r$, along cutdim cd

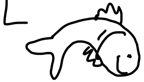


r.leftPart(cd, s)

→ returns rect with low = r.low + high = r.high but high[cd] ← s[cd]

r.rightPart(cd, s)

→ high = r.high + low = r.low but low[cd] ← s[cd]

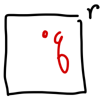


Useful methods:

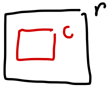
Let r, c - Rectangle

q - Point

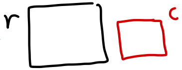
r.contains(q)



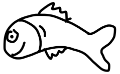
r.contains(c)



r.isDisjointFrom(c)



Kd-Tree Queries II



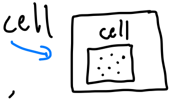
Orthog. Range Query

- Assume: Each node p stores:
  - **p.pt**: splitting point
  - **p.cutDim**: cutting dim
  - **p.size**: no. of pts in p's subtree
- Tree stores ptr. to **root** and **bounding box** for all pts.
- Recursive helper stores current **node p** + **p's cell**.

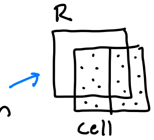
Cases:

- p == null → fell out of tree → 0
- Query rect is **disjoint** from p's cell
  → return 0
  → no point of p contributes to answer

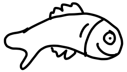
- Query rect **contains** p's cell
  → return p.size
  → every point of p's subtree contributes to answer.

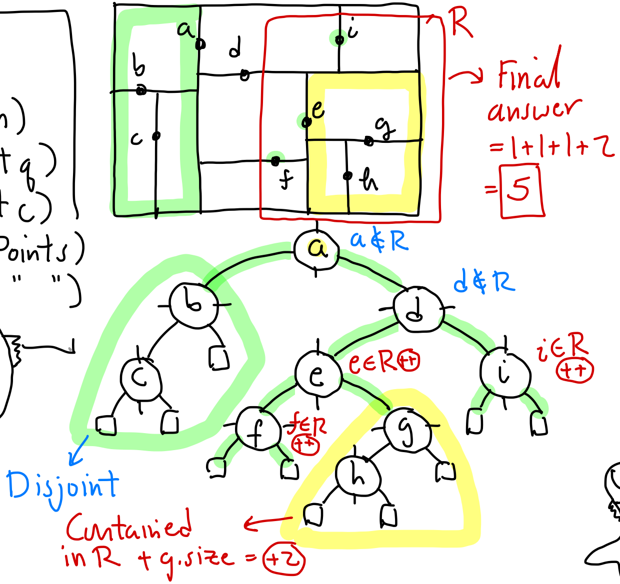


- Otherwise: Rect. + cell **overlap** – Recurse on both children



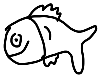
```
class Rectangle {
    private Point low, high
    public Rect(Point l, Point h)
    "    boolean contains(Point q)
    "    boolean contains(Rect c)
    "    Rect leftPart(int cd, Point s)
    "    Rect rightPart("  "    "  ")
}
```





```
int rangeCount(Rect R, KDNode p, Rect cell)
    if (p == null) return 0  // fell out of tree
    else if (R.isDijointFrom(cell)) return 0  // no overlap
    else if (R.contains(cell)) return p.size  // take all
    else { int ct = 0
        if (R.contains(p.pt)) ct++  // p's pt in range
        ct += rangeCount(R, p.left,
                         cell.leftPart(p.cutDim, p.pt))
        ct += rangeCount(R, p.right, cell.rightPart...
    }
```

Kd-Tree Queries III

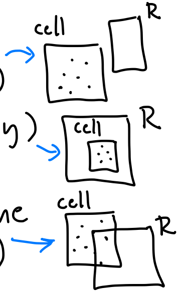


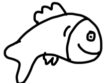
Theorem: Given a balanced kd-tree storing n pts in $\mathbb{R}^2$ (using alternating cut dim), orthog. range queries can be answered in $O(\sqrt{n})$ time.

→ Slower than log n. Faster than n

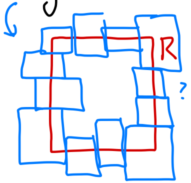
Stabbing: 3 cases

- cell is disjoint (easy)
- cell is contained (easy)
- cell partially overlaps or is stabbed by the query range (hard!)

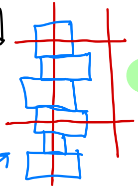




How many cells are stabbed by R? (worst case)



Simpler: Extend R's sides to 4 lines + analyze each one.



Analysis: How efficient is our algorithm?

→ Tricky to analyze

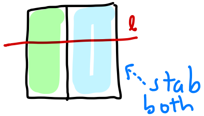
→ At some nodes we recurse on both children ⇒ $O(n)$ time?

→ At some we don't recurse at all!

Lemma: Given a kd-tree (as in Thm above) and horiz. or vert. line ℓ, at most $O(\sqrt{n})$ cells can be stabbed by ℓ

Proof: w.l.o.g. ℓ is horiz.

Cases: p splits vertically



stab both

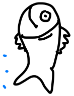
Solving the Recurrence:

- **Macho:** Expand it
- **Wimpy:** Master Thm (CLRS)

Master Thm: $T(n) = aT(n/b) + n^d$ + $d < \log_b a$

$\Rightarrow T(n) = n^{\log_b a}$

For us: $a=2$, $b=4$, $d=0$ $\Rightarrow T(n) = n^{\log_4 2} = n^{1/2} = \sqrt{n}$



Since tree is balanced a child has half the pts + grandchild has quarter.

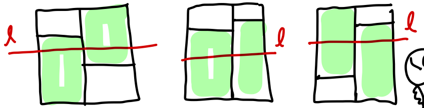
Recurrence: $T(n) = 2 + 2T(n/4)$

2 cells stabbed

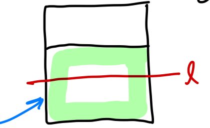
Recurse on 2 grandchildren

Each has n/4 pts

If we consider 2 consecutive levels of kd-tree, ℓ stabs at most 2 of 4 cells:



p splits horizontally

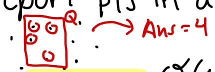


ℓ stabs only one

Range Trees I

Recap:

- **kd-Tree:** General-purpose data structure for pts in $\mathbb{R}^d$
- **Orthogonal range query:** Count/report pts in axis-aligned rect.



- kd-Tree: **Counting:** $O(\sqrt{n})$ time
  Report: $O(k+\sqrt{n})$ time

No. of pts reported

Can we do better?

Range Trees:

- Space is $O(n \log^{d-1} n)$
- Query time:
  - **Counting:** $O(\log^d n)$
  - **Reporting:** $O(k + \log^d n)$

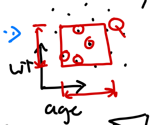
→ In $\mathbb{R}^2$: $\log^2 n$ **much better** than $\sqrt{n}$ for large n

→ Range trees are **more limited**

Layering:

Combing search structures

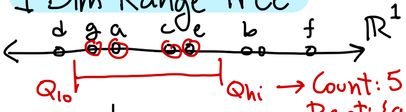
- Suppose you want to answer a **composite query** w. multiple criteria:
- Medical data: Count subjects w.
  - **Age range:** $a_{lo} \leq age \leq a_{hi}$
  - **Weight range:** $w_{lo} \leq weight \leq w_{hi}$



- Design a data structure for each criterion **individually**
- **Layer** these structures together to answer full query

→ **Multi-Layer Data Structures**

1-Dim Range Tree:



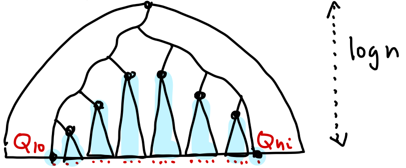
→ Count: 5
Rept: {g, a, c, e}

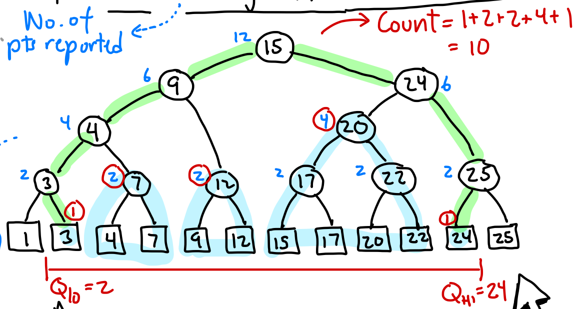
Approach:

- Balanced BST (eg. AVL, RB, ..)
- Assume **extended tree**
- Each node p stores no. of entrie in subtree: **p.size**

Canonical Subsets:

- **Goal:** Express answer as disjoint union of subsets
- **Method:** Search for Q_{lo} + Q_{hi} + take maximal subtrees





Call this a **1-Dim Range Tree:**

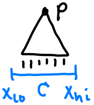
Claim: A 1-Dim range tree with n pts has space $O(n)$ and answers 1-D range count/rept queries in time $O(\log n)$ (or $O(k + \log n)$)

Range Trees II

More details:

Given a 1-D range tree T:

- Let $Q = [Q_{lo}, Q_{hi}]$ be **query interval**
- For each node p, define interval **cell** $C = [x_0, x_1]$ s.t. all pts of p's subtree lie in C
- **Root cell**: $C_0 = [-\infty, +\infty]$



Recursive helper:

int range1Dx(Node p,
Intv $Q = [Q_{lo}, Q_{hi}]$, Intv $C = [x_0, x_1]$)

initial call: range1Dx(root, Q, C_0)

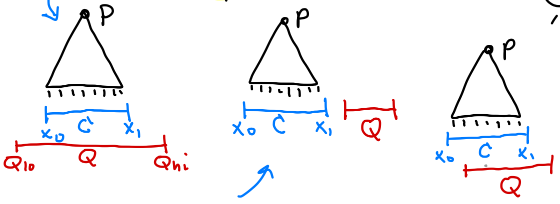
Cases:

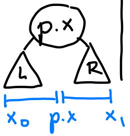
p is external:

- if p.pt.x ∈ Q → 1 else → 0

p is internal:

- **C ⊆ Q** ⇒ all of p's pts lie within query
  → return **p.size**
- **C is disjoint from Q** ⇒ none of p's pts lie in Q
  → return **0**
- Else **partial overlap**
  → Recurse on p's children + trim the cell





```
int range1Dx(Node p,
    Intv Q, Intv C = [x0, x1]) {
  if (p is external)
    return p.pt.x ∈ Q   --T--> 1   --F--> 0
  else if (C ⊆ Q) return p.size
  else if (Q + C disjoint) return 0
  else return:
      range1Dx(p.left, Q, [x0, p.x])
    + range1Dx(p.right, Q, [p.x, x1])
```

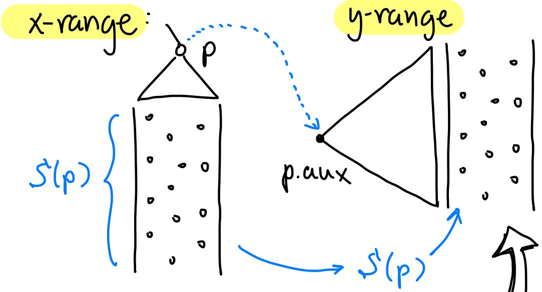
Analysis:

Lemma: Given a 1-D range tree with n pts, given any interval Q, can compute $O(\log n)$ subtrees whose union is answer to query.

Thm: Given 1-D range tree... can answer range queries in time $O(\log n)$ ....→ (+k to report)

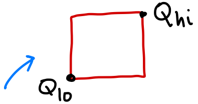
2-D Range Searching:

- "**Layer**" a range tree for x with range tree for y
- For each node p ∈ 1D-x tree, let **S(p)** = set of pts in p's subtree
- Def: **p.aux**: A 1D-y tree for S(p)



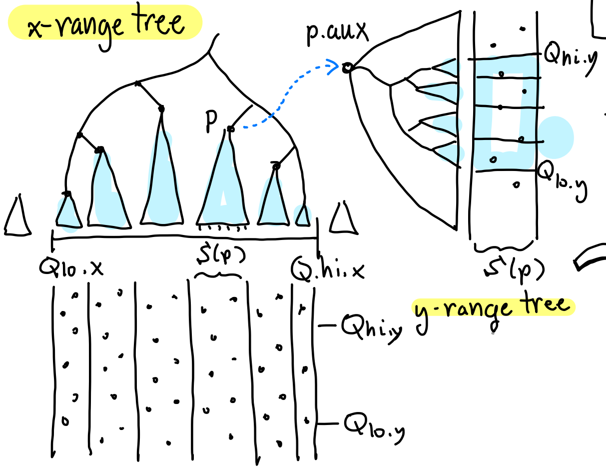
Range Trees III

Answering Queries?



Given query range
$Q=[Q_{lo}.x, Q_{hi}.x]\times[Q_{lo}.y, Q_{hi}.y]$

- Run range1Dx to find all subtrees that contribute
  - For each such node p, run range1Dy on p.aux
- Return sum of all result



Intuition: The x-layer finds subtrees p contained in x-range + each aux tree filters based on y.

2D Range Tree:

- Construct 1D range tree based on x coords for all pts
- For each node p:
  - Let S(p) be pts of p's tree
  - Build 1D range tree for S(p) based on y → p.aux
- Final structure is union of x-tree + (n-1) y-trees

Higher Dimensions?

- In d-dim space, we create d-layers
- Each recurses one dim lower until we reach 1-d search
- Time is the product:

$$\underbrace{\log n \cdot \log n \cdots \log n}_{d} = O(\log^d n)$$

Analysis: The 1D x search takes of O(log n) time + generates O(log n) calls to 1D y search

⇒ Total: $O(\log n \cdot \log n) = O(\log^2 n)$

```
int range2D(Node p, Rect Q, Intv C=[$x_0$,$x_1$]){
    if (p is external) return p.pt ∈ Q? $\xrightarrow{T}$ 1 $\xrightarrow{F}$ 0
    else if (Q.x contains C){        // C ⊆ Q's x-projection
        [$y_0$,$y_1$] = [-∞,+∞]        // init y-cell
        return range1Dy(p.aux, Q, [$y_0$,$y_1$])
    } else if (Q.x is disjoint of C) return 0
    else                             // partial x-overlap
        return range2D(p.left, Q, [$x_0$, p.x])
             + range2D(p.right, Q, [p.x, $x_1$])
}
```

Analysis:

Invoked O(log n) times - once per maximal subtree

Invoked O(log n) times - once for each ancestor of max subtree

History:

- Driscoll, Sarnak, Sleator, Tarjan (1986) - First serious theoretical analysis

Splay trees

→ Applied to geometric search (time is coordinate)

Approaches:

Copy-on-write: Whenever you modify, make a copy

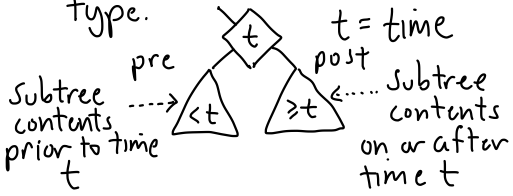
→ very inefficient!

Change-log: Make a list of recent updates

→ slow to process

Fat nodes / Node copying: When changes occur, save just modified portions

→ **Temporal node:** New node type.



Persistent Data Structures:

- Preserves prior states of data structure
- Allows searches in history
  → "Was Jane Smith enrolled in UMD in Spring 2016?"
- Full/Partial Persistence:
  - change can be made to any/current version

Persistent Search Trees I

Case study: PBJ Tree

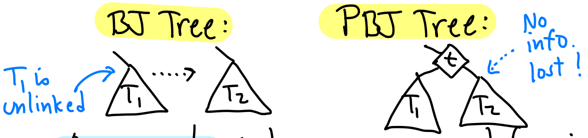
Persistent Weight-Balanced Jackhammer trees

→ A partially persistent BJ tree

→ Uses rebuilding to balance subtrees

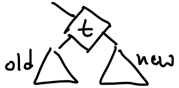
→ Allows weighted entries

Example: Rebuild subtree T_1 at time t. Let T_2 be new subtree:



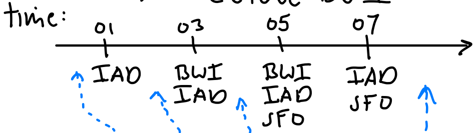
Approach: Whenever a modification made - save old contents + use temporal node to distinguish

Change at t:



Example:

Time:	Command:
01	insert IAD
03	insert BWI
05	insert SFO
07	delete BWI



find IAD at 00: not found
find IAD at 02: found
get-min at 04: BWI
get-min at 08: IAD

Persistent Search Trees II

PBJ Tree: Tech Specs

Node structure:

- **Node**: weight + maxWt (float)
  - ↳ **TempNode**: time (int)
    pre, post (Node)
  - **IntNode**: key, left, right
  - **ExtNode**: key, value

PBJ Tree (Private) Data:

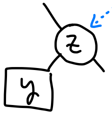
- final float **ALPHA, BETA**
  ↳ same as BJ Tree
- Node **root** - root → init: null
- int **firstInsertTime**
- int **lastInsertTime**
  } → init: -1

Insertion (without rebalancing)

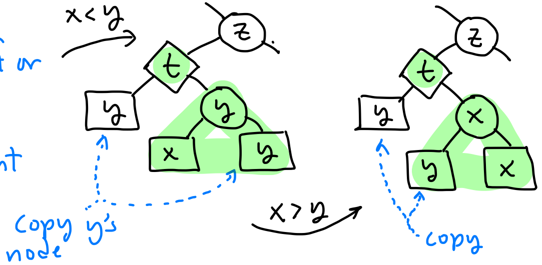
Helper: **Node insert(x, v, w, t)**

→ insert (x,v) of wt w at t

- If root == null: **(first insertion)**
  - firstInsertTime ← t
  - root = new ExtNode (x, v, w)
- else
  - search for x in tree
    - Int: left ← left.insert(...)
      right ← right.insert(...)
    - Temp: post ← post.insert(...)
  - reach ExtNode $\boxed{y}$
    - if x == y ⇒ Error: Duplicate key
    - else:



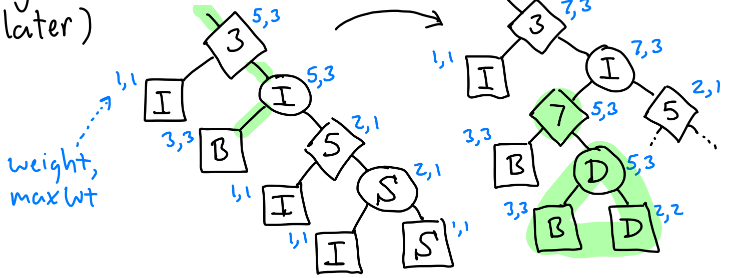
Parent of y
- May be Int or Temp
- $\boxed{y}$ may be left or right (or post) child



→ Update node weight + maxWt (later)

Example:

insert("D") wt=2 time=7



Updating Weights?

- **Int Node**: wt ← left.wt + right.wt
  maxWt ← max(left.maxWt, right...)
- **Temp Node**: wt ← post.wt ← only post!
  maxWt ← post.maxWt

Few More Things:

- Before insert, check that
  t > lastInsertTime
  → else Error!
- Set **lastInsertTime ← t**
- Note: Partial persistence
- **Rebalance** → Next

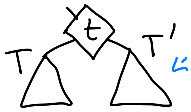
Persistent Search Trees III

Rebalancing after insertion:

- Starting at root, retrace search path
- Recursive helper:
  Node rebalance(x,t)
  → **Internal:** Apply to left/right based on x ↔ key
  → **Temporal:** Apply to post only.

Internal Node Rebalance:

- Test α,β condition (same as BJ tree)
- If unbalanced, compile list A of extern nodes for **current tree** ← For temporal recurse only on post side
- T = original tree
- T' = buildTree(A)
- return:



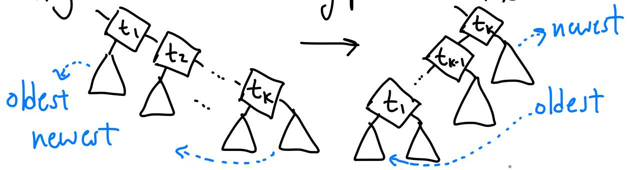
The new tree has no temporal nodes

Temporal Node Rebalance:

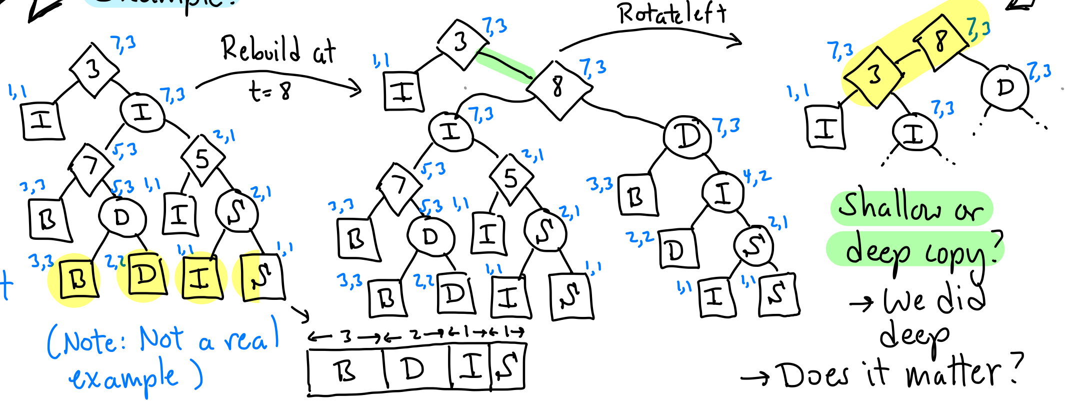
- Recurse on post side:
  post = post.rebalance(x,t)
- On return:
  if (post child is temporal)...
  - perform left rotation
  - update weights

  return root of subtree

Why? Don't like long post chains



Example:



(Note: Not a real example)

find(x,t), findUp(x,t), getMin(t)...

- Same as before but for temporal nodes visit pre/post based on t
- check whether t < firstUpdateTime
  → if so → null

getPreorderList(t)/getFullPreorderList()

- First gets preorder list at time t (no temporal nodes!)
- Second gets full preorder list (all nodes)

Delete/Clear: Not implemented/Not required! (A bit messy)

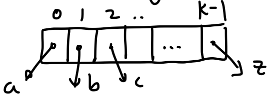
Shallow or deep copy?
→ We did deep
→ Does it matter?

Tries and Digital Search Trees I

Tries: History

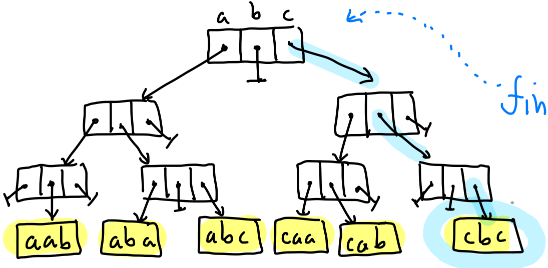
- de la Briandais (1959)
- Fredkin - "trie" from "retrieval"
- Pronounced like "try"

Node: Multiway of order k

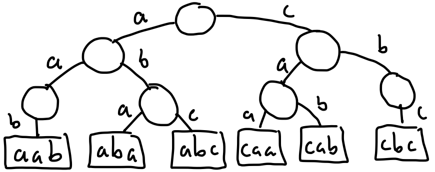


Example: $\Sigma = \{a=0, b=1, c=2\}$

Keys: {aab, aba, abc, caa, cab, cbc}

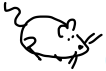


Same structure/Alt. Drawing



Digital Search:

- Keys are strings over some alphabet Σ
- E.g. $\Sigma = \{a, b, c, \ldots\}$
  $\Sigma = \{0, 1\}$ Let $k = |\Sigma|$
- Assume chars coded as ints: a=0, b=1, ... z=k-1



Analysis:

Search: ~ length of query string [O(1) time per node]

Space:

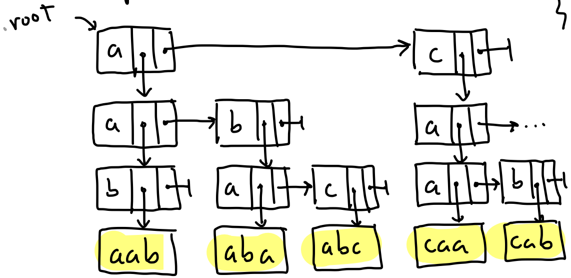
- No. of nodes ~ total no. of chars in all strings
- Space ~ k·(no. of nodes)

Large!

Analysis:

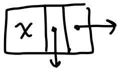
- **Space**: Smaller by factor k
- **Search Time**: Larger by factor of k

Example:



How to save space?

de la Briandais trees:

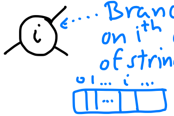
- Store 1 char. per node
- 

  ≠ x ⇒ try next char in Σ
  == x ⇒ advance to next character of search string
- First-child/next-sibling

Tries and Digital Search Trees II

Patricia Tries:

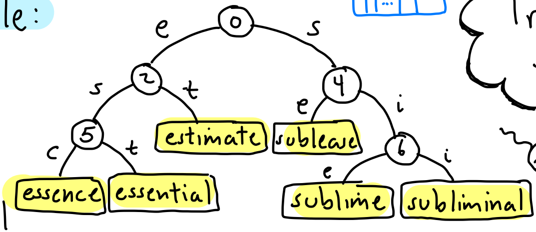
- Improves trie by compressing degenerate paths
- PATRICIA = Practical Alg. to Retrieve Info. Coded in Alpha...
- Late 1960's: Morrison + Gwehenberger
- Each node has index field, indicates which char to check next (Increase with depth)



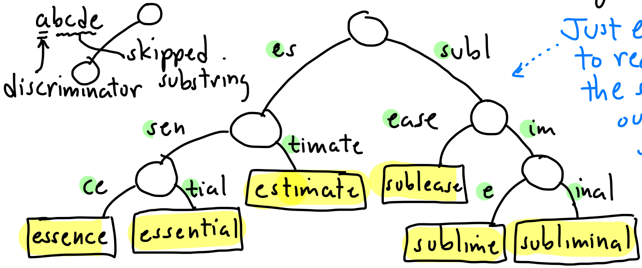
Branch based on ith char of string

Example:

0 1 2 3 ...
essence
essential
estimate
sublease
sublime
subliminal



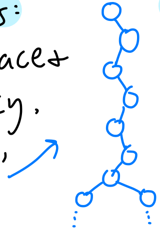
Same data structure – Drawn differently



Just easier to read the strings out.... Same data struct.

Dealing with long Paths:

- To get both good space + query time efficiency, need to avoid long, degenerate paths.



- Path compression!

Example:

	ID
S_{10} : $	$
S_9 : a$	a$
S_8 : ma$	ma$
S_7 : ama$	ama$
S_6 : jama$	j
S_5 : ajam...	aj
S_4 : pajam...	paj
S_3 : apaja...	ap
S_2 : mapaj...	map
S_1 : amapaj...	amap
S_0 : pamapa...	pam

Example: S = pamapajama$

0 1 2 3 4 5 6 7 8 9 10

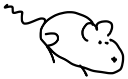
S_{10} - $
S_9 - a$
S_8 - ma$
S_7 - ama$
⋮

Def: Substring identifier for S_i is shortest prefix of S_i unique to this string

E.g. ID(S_1) = "amap"
ID(S_7) = "ama$"

Analysis:

- Query time: (Same as std trie) ~ search string length (may be less)
- Space:
  - No. nodes: ~ No. of strings (irresp. of length)
  - Total space: k·(No. of nodes) + (Storage for strings)



Suffix Trees:

- Given single large text S
- Substring queries: "How many occurrences of "tree" in CMSC 420 notes"

Notation: $S = a_0 a_1 a_2 \ldots a_{n-1}\$$

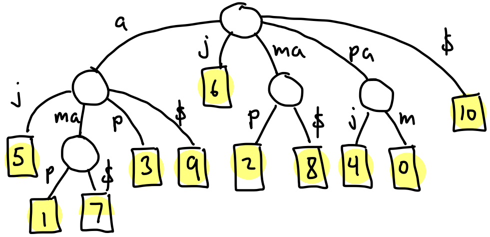
special terminal

- Suffix: $S_i = a_i a_{i+1} \ldots a_{n-1}\$$
- Q: What is minimum substring needed to identify suffix S_i?

Tries and Digital Search Trees III

0 1 2 3 4 5 6 7 8 9 10

Example: S = pamapajama\$



E.g. $ID(S_1)$ = amap $ID(S_7)$ = ama\$

Substring Queries:

How many occurrences of t in text?

- Search for target string t in trie
- if we end in internal node (or midway on edge) – return no. of extern. nodes in this subtree
- else (fall of on extern node)
  - compare target with string
    - if matches – found 1 occurrence
    - else – no occurrences

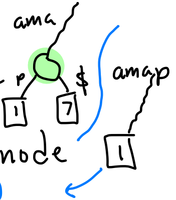
Example:

Search("ama") → End at intern node

Report: 2 occ's.

Search("amapaj") → End at extern node

Go to S_1 + verify



Suffix Trees (cont.)

S - text string $|S| = n$

S_i = i^{th} suffix

Substring ID = min substr. needed to identify S_i

A **suffix tree** is a Patricia trie of the n+1 substring identifiers

Analysis:

- **Space:** O(n) nodes
  O(n·k) total space
  (k = |Σ| = O(1))
- **Search time:** ~ to length of target string
- **Construction time:**
  - O(n·k) [nontrivial]

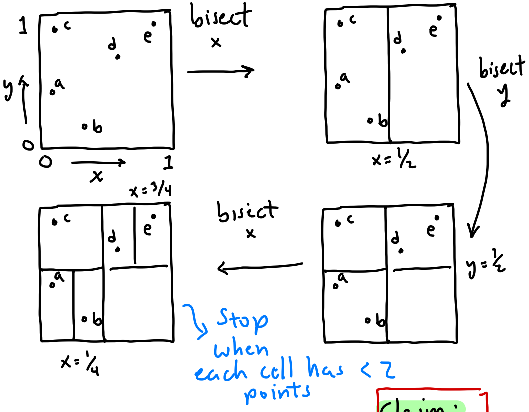
PR k-d tree:

Can be used for answering same queries as point kd-tree (orth. range, near. neigh)

Geometric Applications:

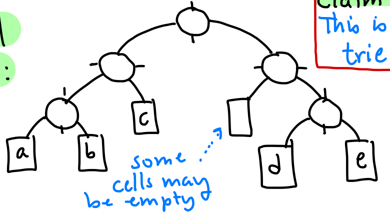
PR kd-Tree: kd-tree based on midpoint subdivision

Assume points lie in unit square



Stop when each cell has < 2 points

Final tree:



Claim: This is a trie!

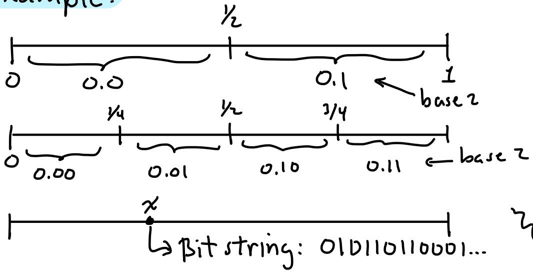
Binary Encoding:

- Assume our points are scaled to lie in unit square $0 \leq x, y < 1$ (can always be done)
- Represent each coordinate as binary fraction:

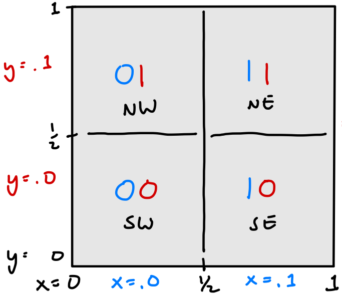
$x = 0.a_1 a_2 a_3 \ldots \quad a_i \in \{0,1\}$

$x = \sum_i a_i \cdot 1/2^i$

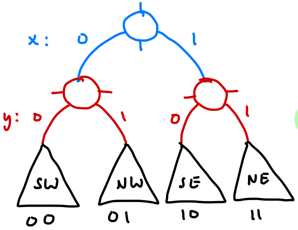
Example:



How do we extend to 2-D?



PR kd-tree



PR kd-Tree ≡ Trie ??

- Approach: Show how to map any point in $\mathbb{R}^2$ to bit string
- Store bit strings in a trie (alphabet $\Sigma = \{0,1\}$)
- Prove that this trie has same structure as kd-tree

Tries and Digital Search Trees IV

Bit Interleaving:

Given a point $p = (x,y)$

$0 \leq x, y < 1$

let: $x = 0.a_1 a_2 \ldots$ in binary

$y = 0.b_1 b_2 \ldots$

Define:

$\phi(x,y) = a_1 b_1 a_2 b_2 a_3 b_3 \ldots$

Called **Morton Code** of p

Further Remarks:

- Techniques for efficiently encoding, building, serializing, compressing... tries apply immediately to PR kd-tree
- Can generalize to any dimension

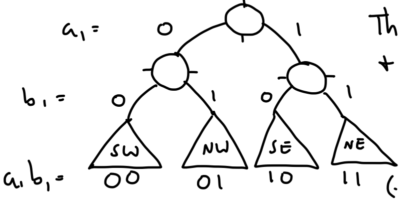
$$\left.\begin{array}{l} x = 0.a_1 a_2 \ldots \\ y = 0.b_1 b_2 \ldots \\ z = 0.c_1 c_2 \ldots \end{array}\right\} \phi = a_1 b_1 c_1 a_2 b_2 c_2 \ldots$$

Lemma: Given a pt set $P \subseteq \mathbb{R}^2$ (in unit square $[0,1)^2$) let $P = \{p_1, \ldots, p_n\}$ where $p_i = (x_i, y_i)$. Let $\Phi(P) = \{\phi(p_1), \phi(p_2), \ldots, \phi(p_n)\}$ (n binary strings). Then the PR kd-tree for P is equivalent to binary trie for $\Phi(P)$.

Proof: By induction on no. of bits

Let $x = 0.a_1 a_2 \ldots \quad y = 0.b_1 b_2 \ldots$

and consider just $\phi(x,y) = a_1 b_1 \ldots$



The PR kd-tree + binary trie assign pts to same subtrees.

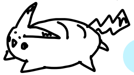
(.... induction)

Memory Management I

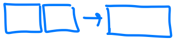
Deallocation Models:

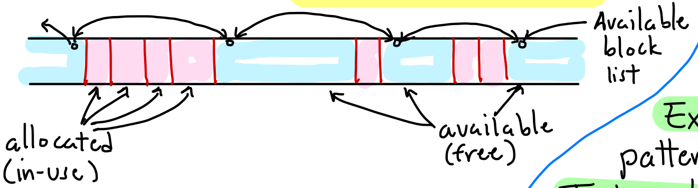
Explicit: (C+C++)
- programmer deletes
- may result in leaks if not careful

Implicit: (Java, Python)
- runtime system deletes
  - Garbage collection
  - Slower runtime
  - Better memory compaction



Explicit Allocation/Deallocation

- Heap memory is split into blocks whenever requests made
- Available blocks:
  - merged when contiguous 
  - stored in available block list



What happens when you do

- new (Java)
- malloc/free (C)
- new/delete (C++) ?

Runtime System Mem. Mgr.

- **Stack** - local vars, recursion
- **Heap** - for "new" objects

Don't confuse with heap data structure / heapsort

Fragmentation:

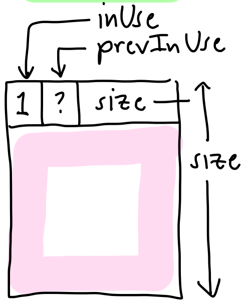
- Results from repeated allocation + deallocation (Swiss-cheese effect)

External: Caused by pattern of alloc/dealloc

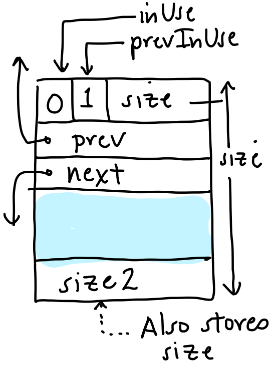
Internal: Induced by mem. manage. policies (not user)

Block Structure:

Allocated:



Available:



Guide:

prevInUse: 1 if prev. contig. block is allocated

prev/next: links in avail. list

size/size2: total block size (includes headers)

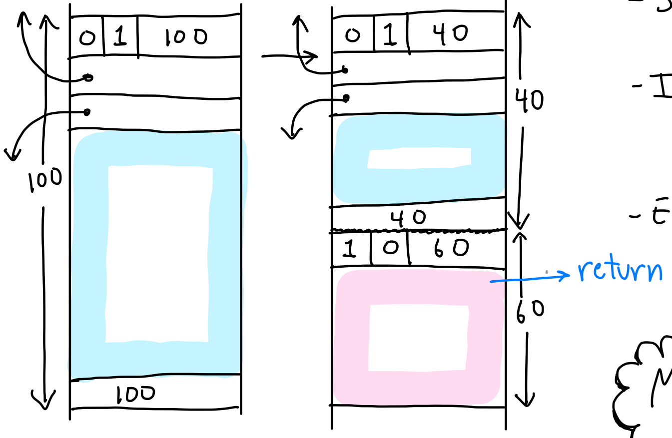
How to Select from available blocks?

- **First-fit:** Take first block from avail. list that is large enough
- **Best fit:** Find closest fit from avail list

Surprise: First-fit is usually better
- faster + avoids small fragments

Memory Management II

Example: Alloc b=59



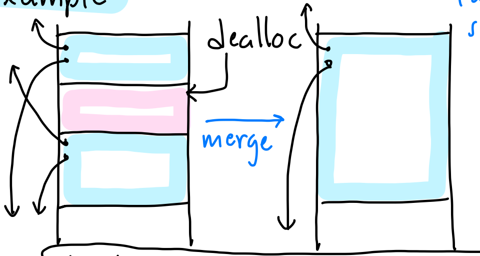
Allocation: malloc(b)

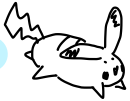
- Search avail. list for block of size b′ ≥ b+1
- If b′ close to b : alloc entire block (unlink from avail list)
- Else : split block

Deallocation:

- If prev + next contiguous blocks are allocated → add this to avail
- Else - merge with either/both to make max. avail block

Example:





Some C-style pointer notation

void* - pointer to generic word of memory

Let p be of type void* :

p+10 - 10 words beyond p

*(p+10) - contents of this

Let p point to head of block:

p.inUse , p.prevInUse, p.size

- we omit bit manipulation

*(p+p.size-1) - references last word in this block

```
(void*) alloc(int b) {
    b += 1                 // add +1 for header
    p = search avail list for block
            size ≥ b
    if ( p == null ) Error-Out of mem!
    if ( p.size - b ≤ TOO_SMALL )
        unlink p from avail. list
        q = p
    else .... (continued)
```

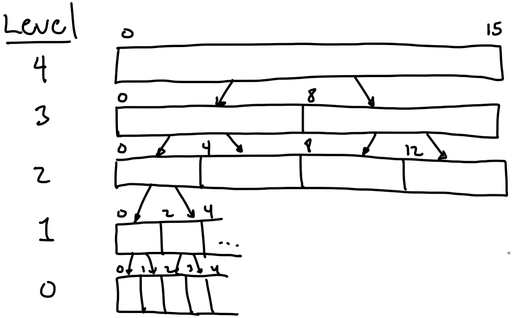
```
else {
    p.size -= b                 // remove allocation
    *(p+p.size-1) = p.size      // size 2
    q = p + p.size              // start of new block
    q.size = b                  } // new block
    q.prevInUse = 0             }    header
}
q.inUse = 1
(q+q.size).prevInUse = 1
            // update prevInUse for next contig. block
return q+1                      // skip over header
}
```

Memory Management III

Buddy System:

- Block sizes (including headers) are power of 2
- Requests are rounded up (internal fragmentation)
- Block size 2^k starts at address that is multiple of 2^k
- k = level of a block

Structure:

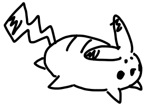


In practice: There is a minimum allowed block size

Buddy system only allows allocations aligning with these blocks

Coping with External Fragmentation

- Unstructured allocation can result in severe external fragmentation
- Can we compress? Problem of pointers
- By adding more structure we can reduce extern frag. at cost of internal frag.



Merging:

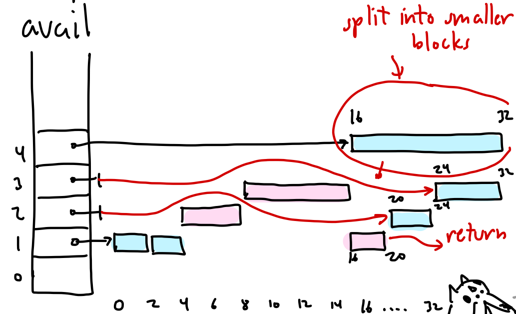
- When two adjacent blocks are available, we don't always merge them
  - → Must have same size: 2^k
  - → Must be buddies - siblings in this tree structure

Def: $\text{buddy}_k(x) = \begin{cases} x + 2^k & \text{if } 2^{k+1} \text{ divides } x \\ x - 2^k & \text{otherwise} \end{cases}$

$\equiv \text{buddy}_k(x) = (1 \ll k) \oplus x$ [Bit manipulation]

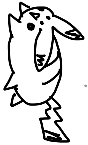
Example: alloc(2) ·······> alloc(4)

round up



Allocation: alloc(b)

- $k = \lceil \lg(b+1) \rceil$ ···> add +1 for header
- if avail[k] non empty - return entry + delete
- else: find avail[j] ≠ ∅ for j > k
  - split this block



Big Picture:

- Avail list is organized by level: avail[k]
- Block header structure same as before except:

prevInUse } not needed
size2