CMSC/Math 456: Cryptography (Fall 2022)

Lecture 22

Daniel Gottesman

Administrative

Problem set #7 is due Thursday at *midnight*.

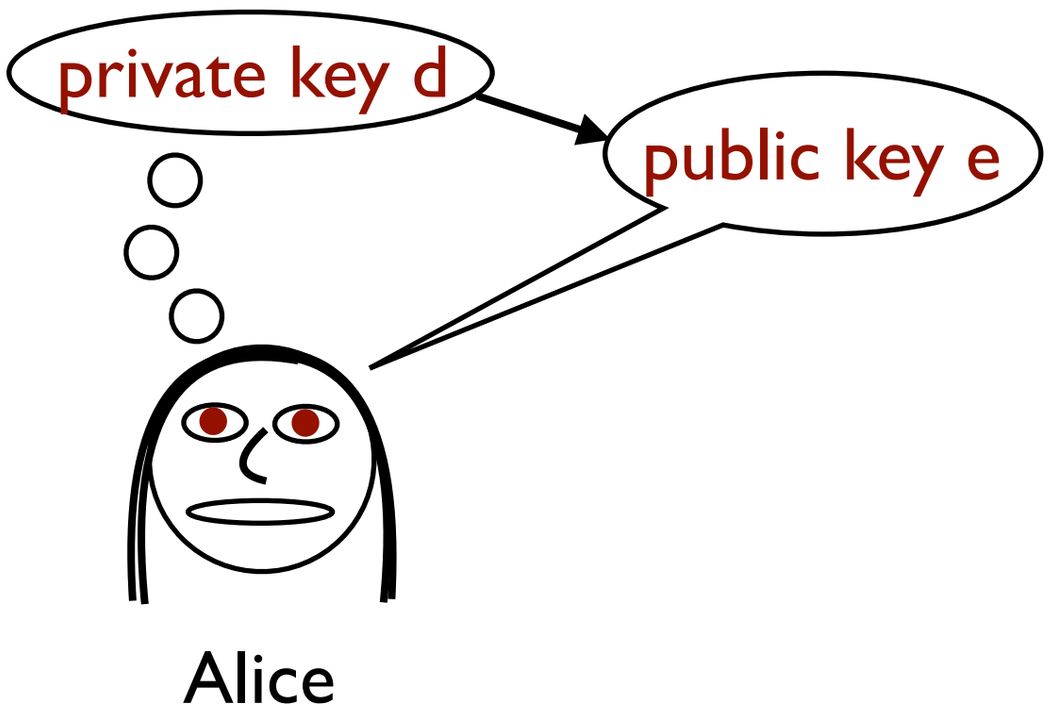
Note: There was an error in eq. (3), now fixed.

Problem set #8 will be assigned on Thursday (Nov. 17) but you will have *two weeks* to do it. (Due Dec. 1)

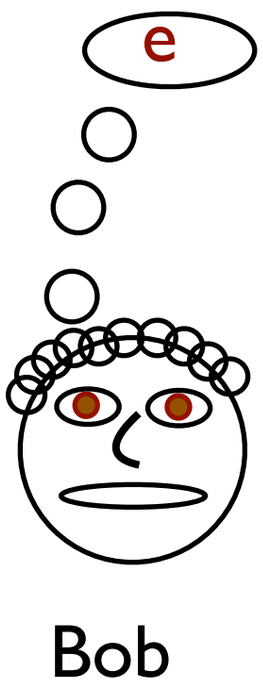
This class is being recorded

Digital Signatures

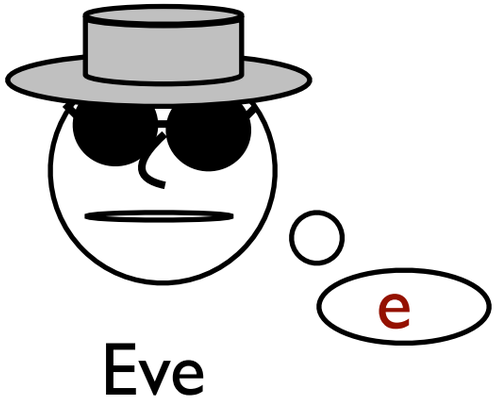
Digital signatures are a public key version of MACs.



Alice



Bob



Eve

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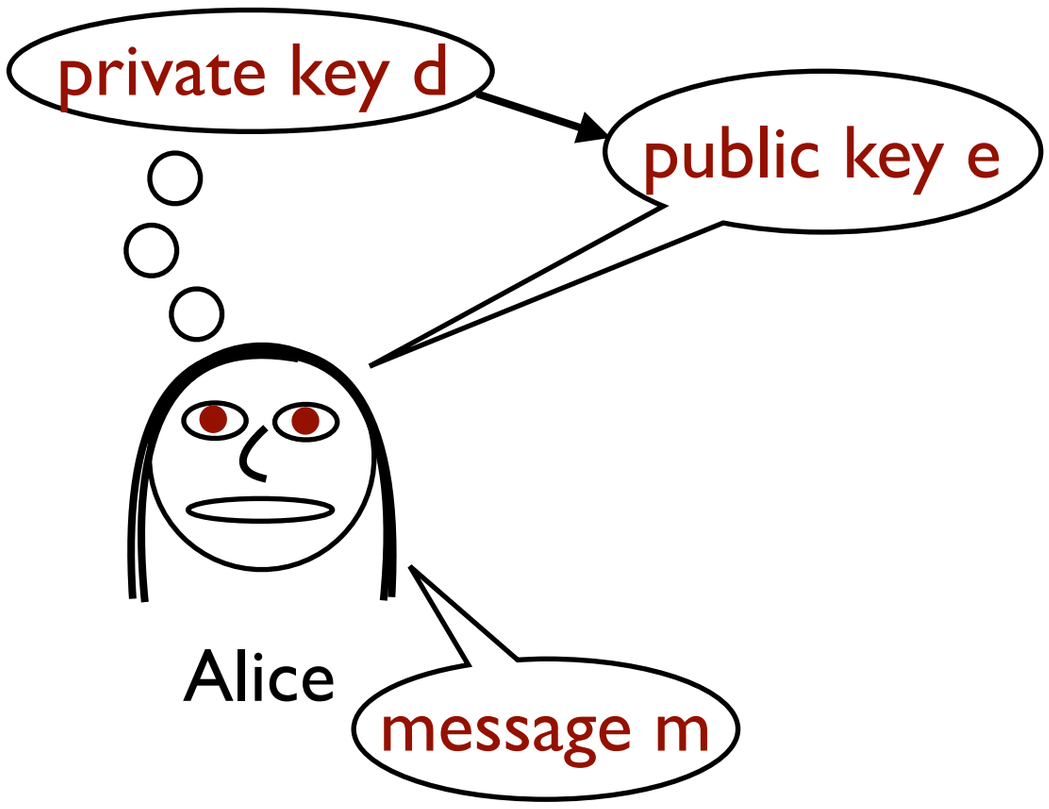

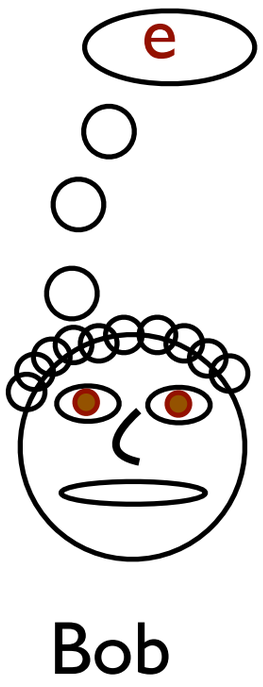

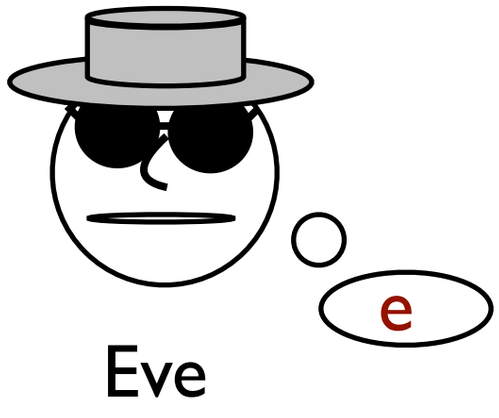

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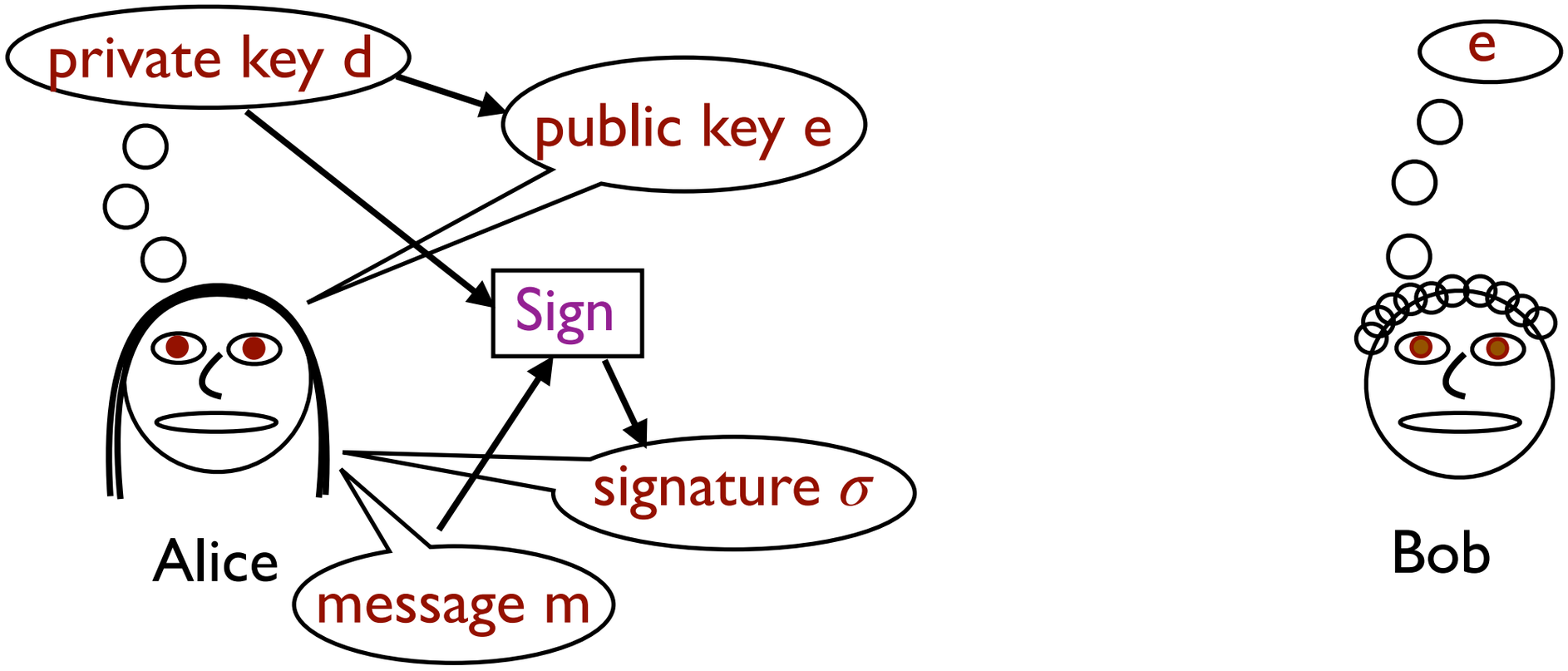

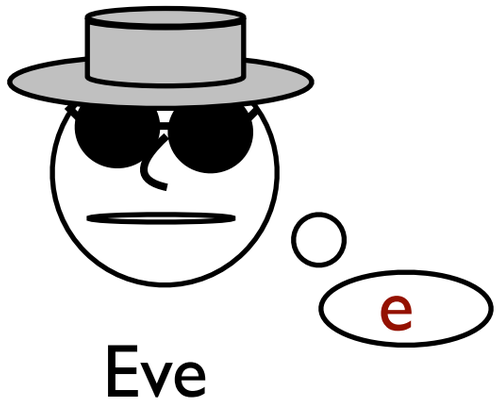

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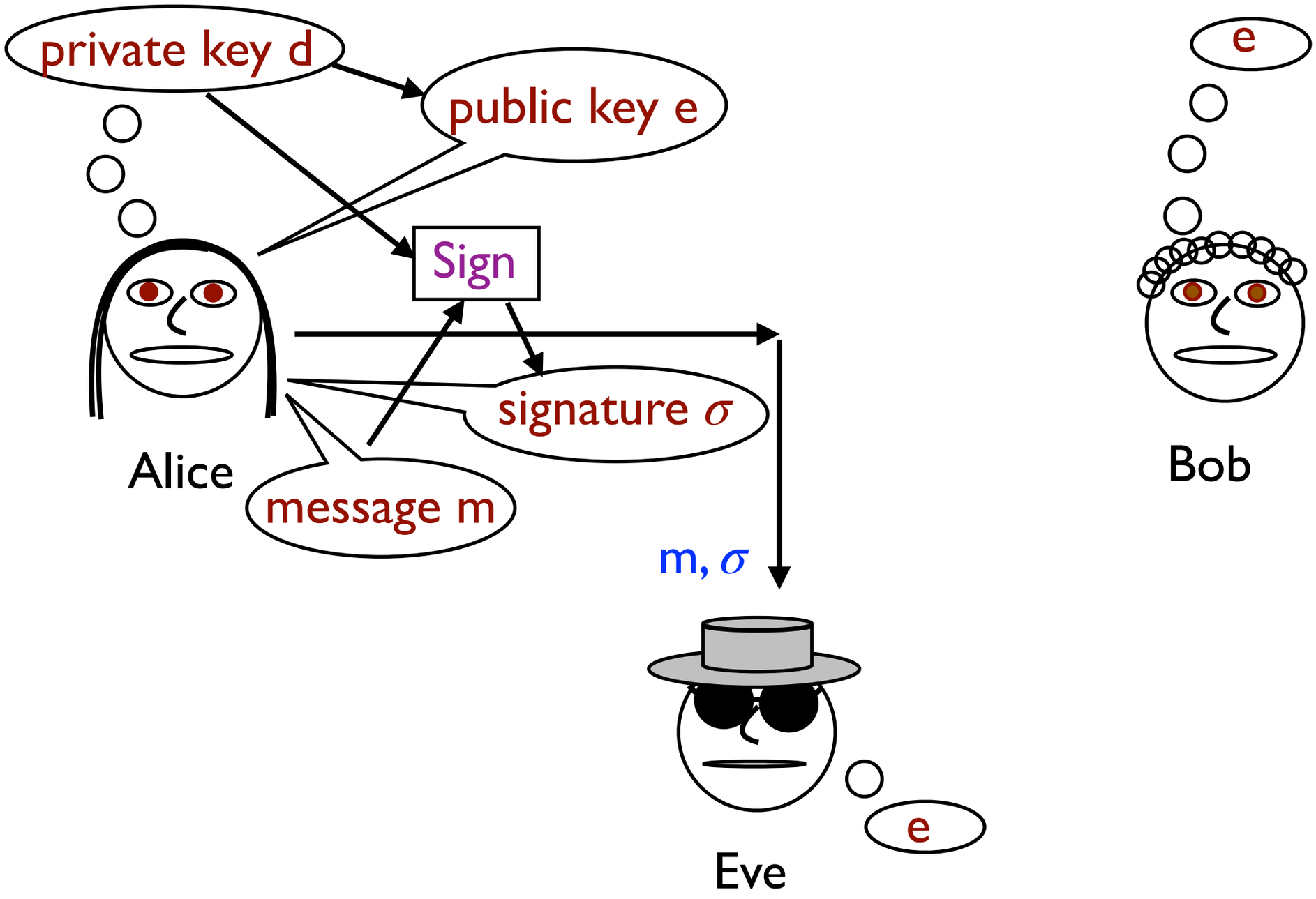

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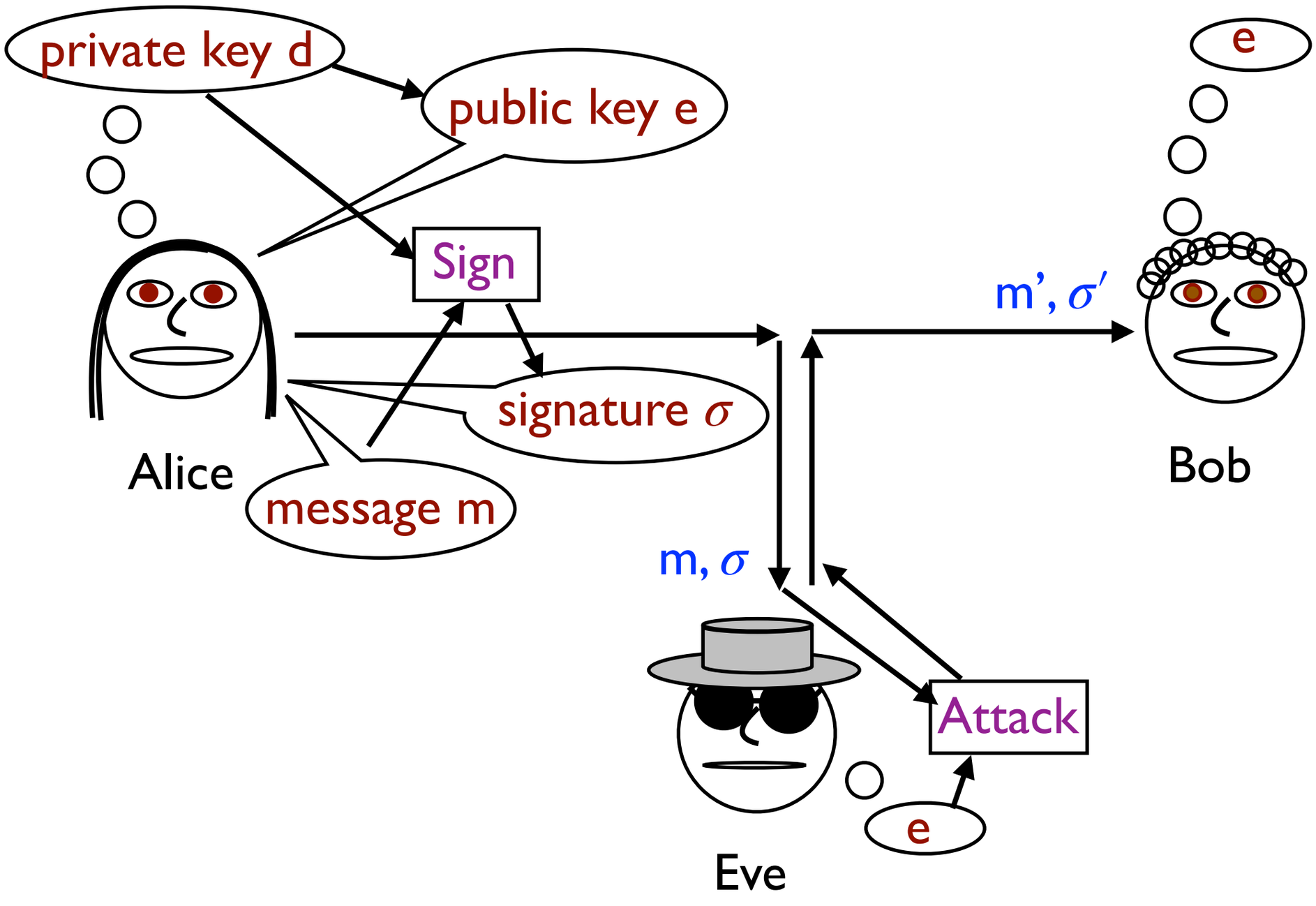

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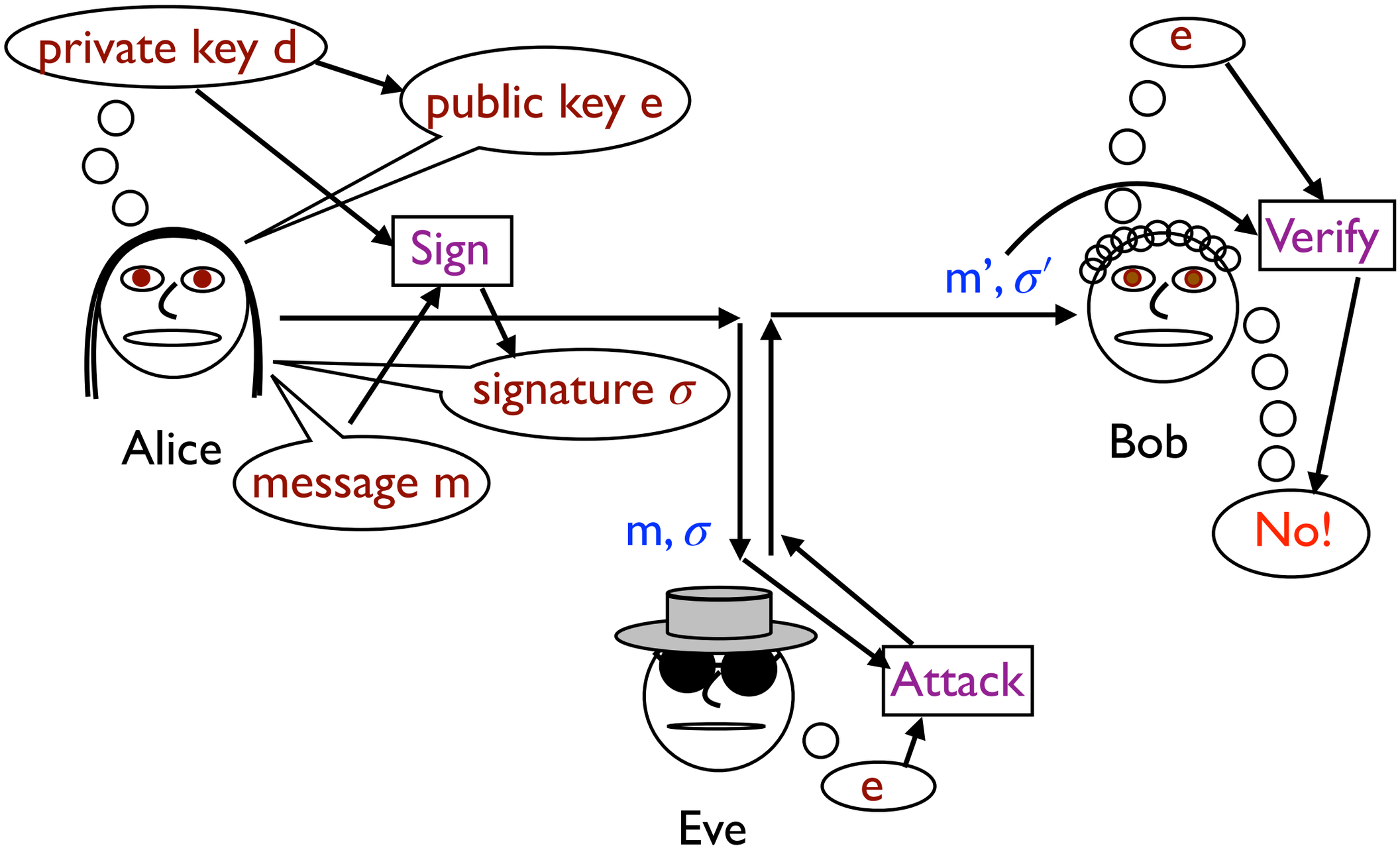

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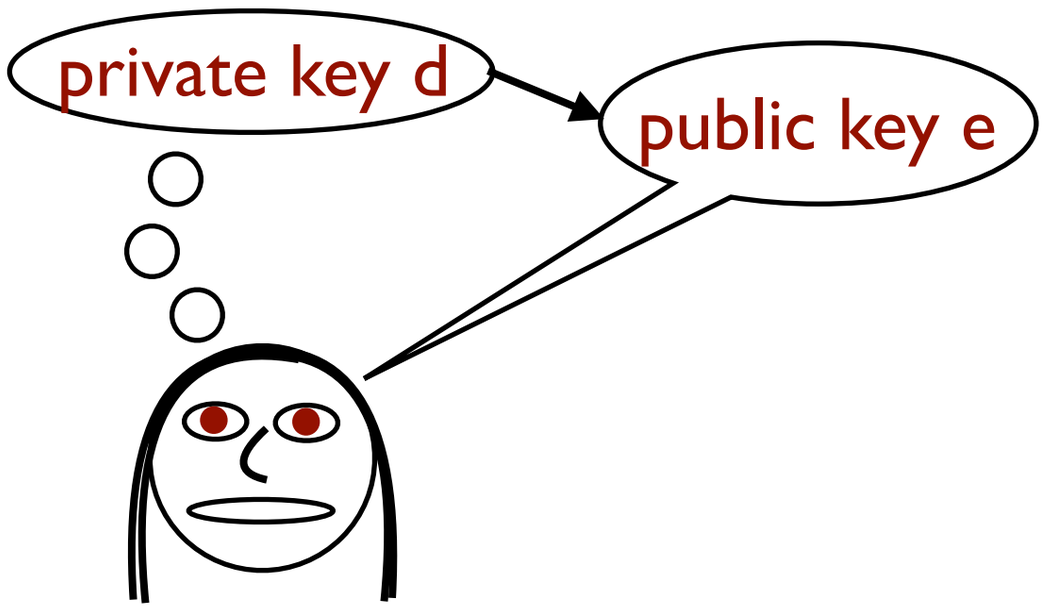
Transferability and Non-Repudiation



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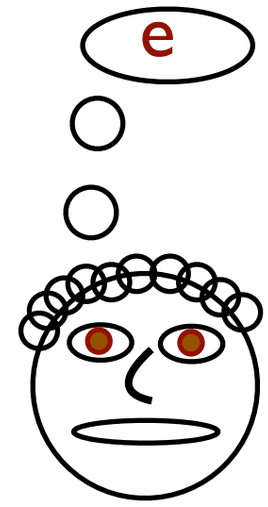

Bob

Because signatures use a public key, they are transferable between recipients. And Alice cannot repudiate the message, claiming she did not send it, since it can only be created using her private key.

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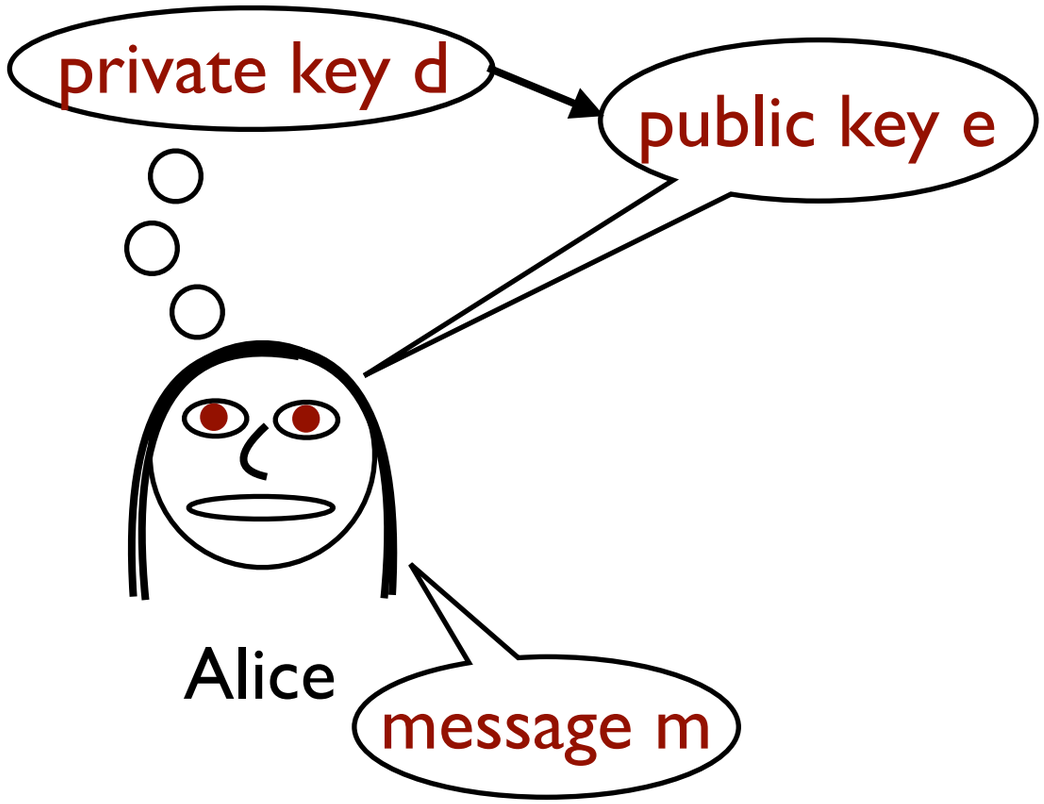

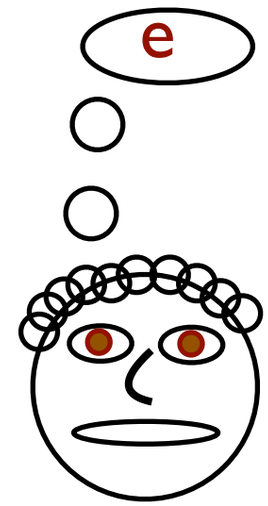

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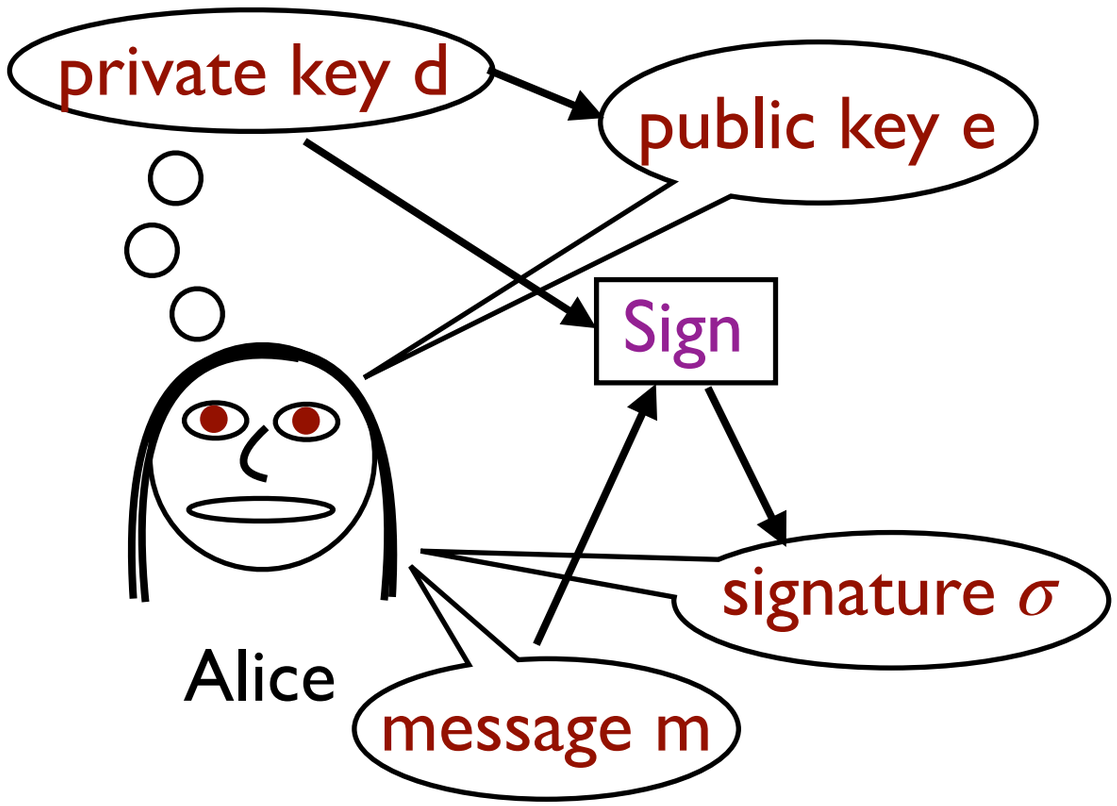

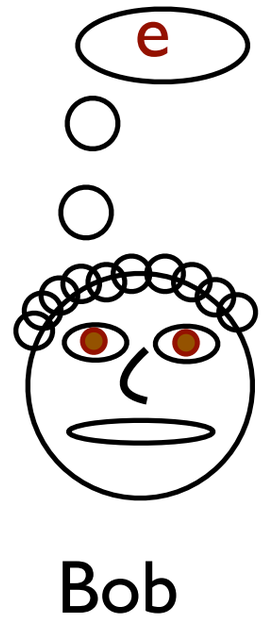

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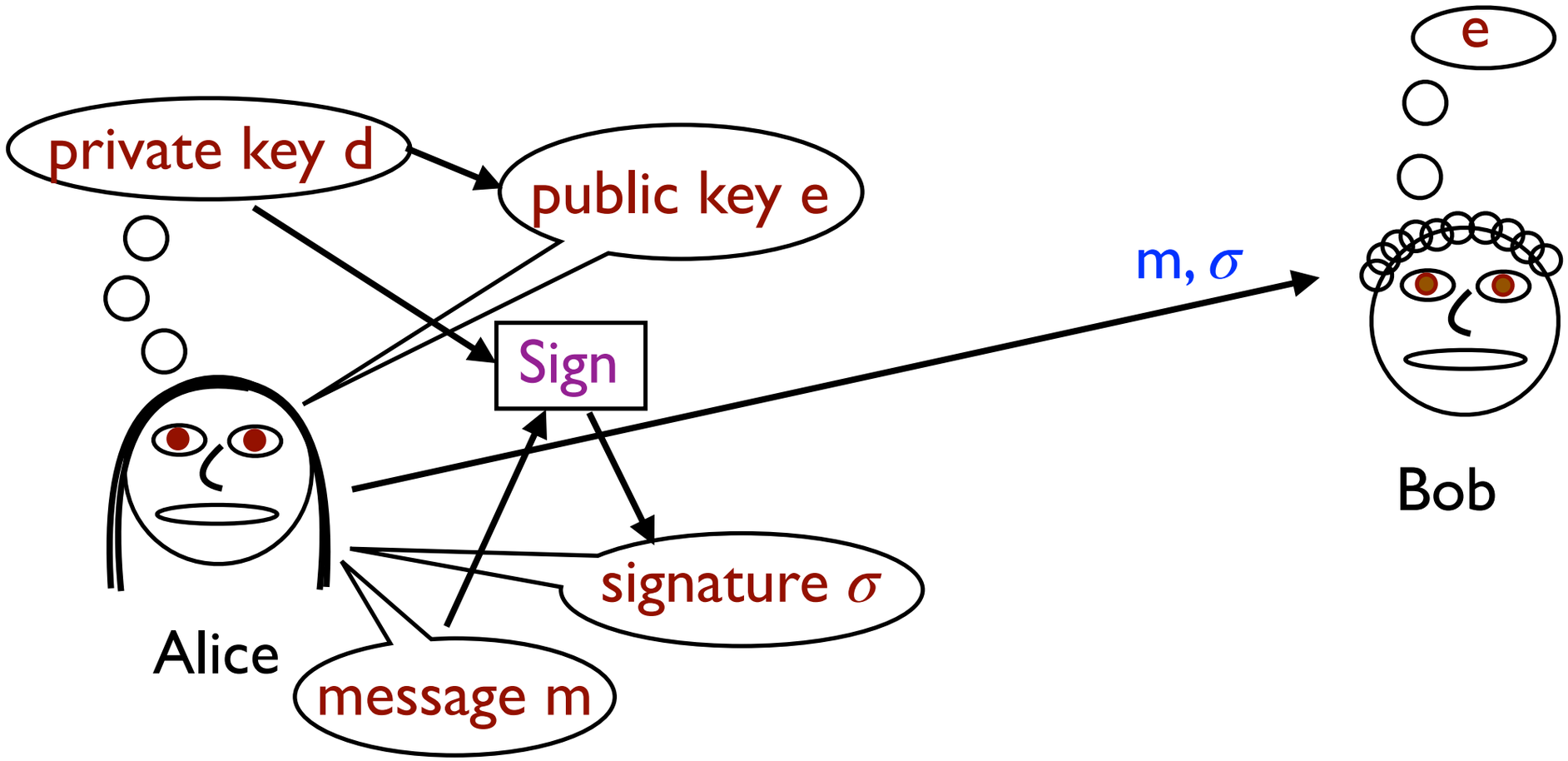

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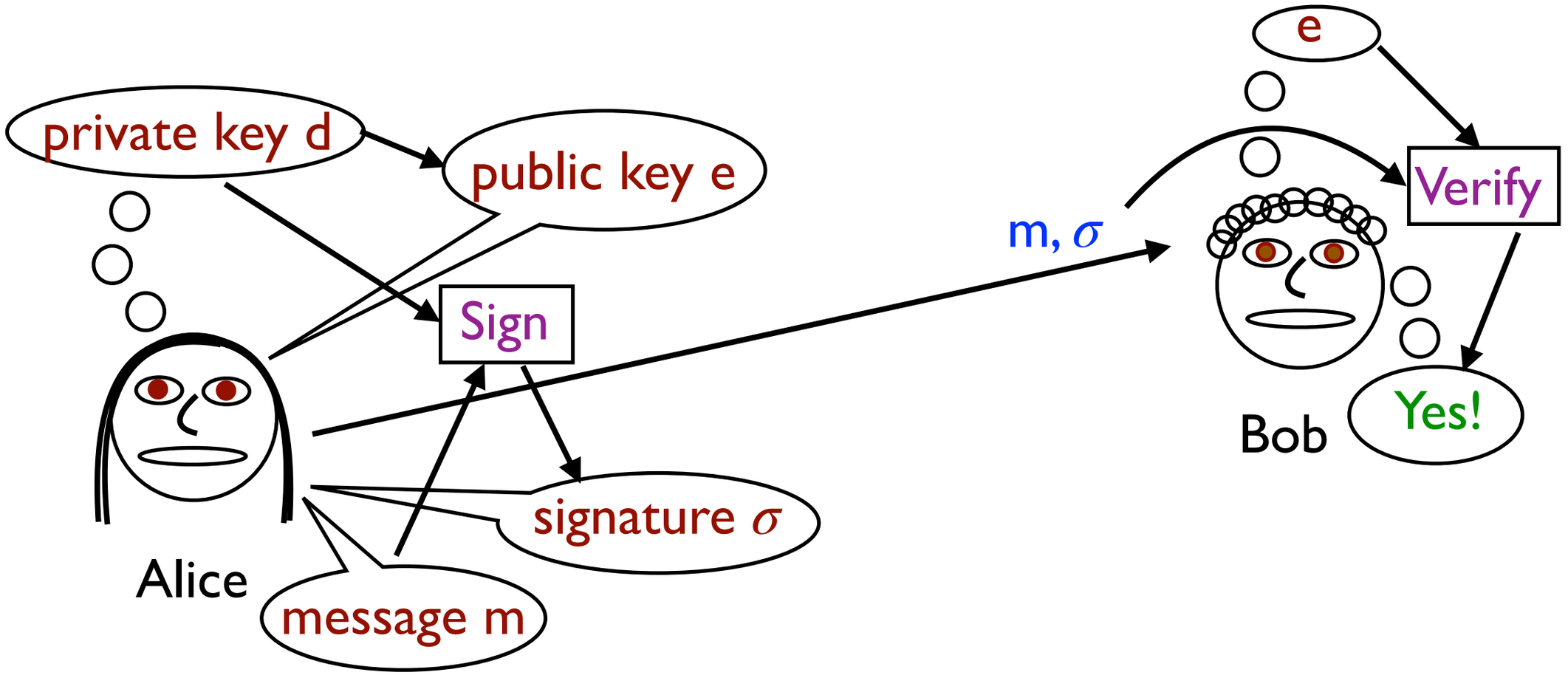

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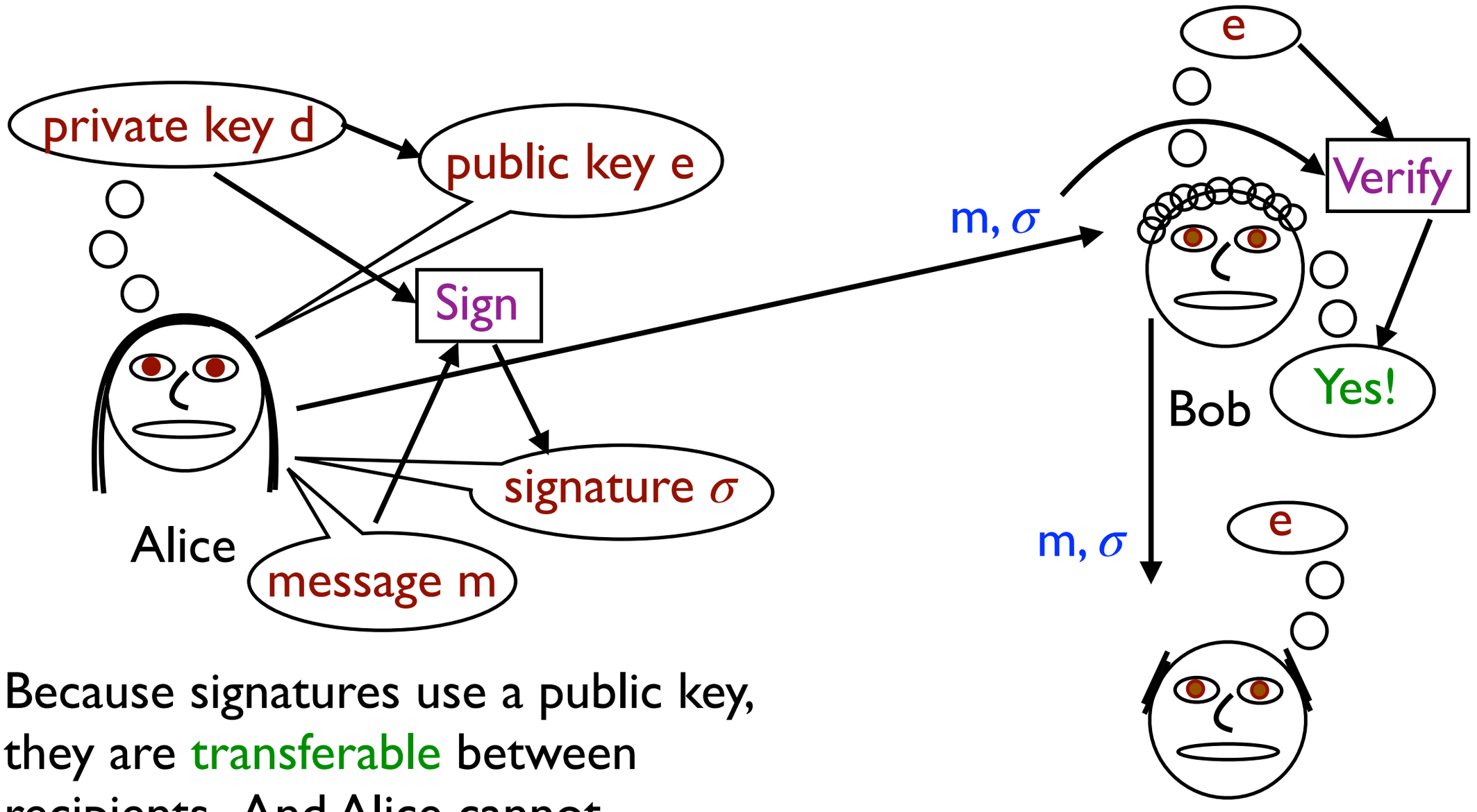

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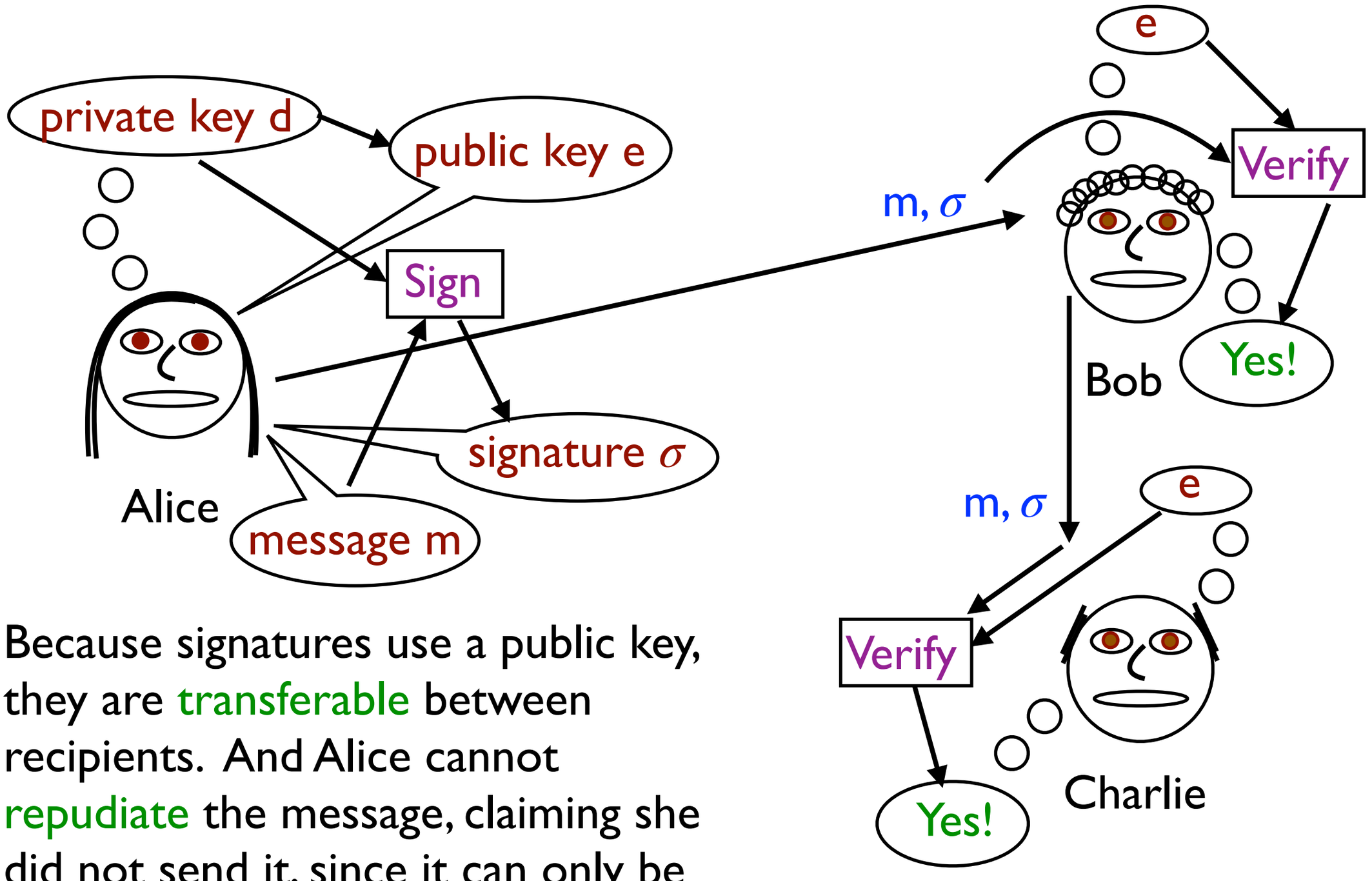

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Digital Signature Definition

Definition: A digital signature is a set of three probabilistic polynomial-time algorithms (Gen, Sign, Vrfy):

Gen is the key generation algorithm. It takes as input s, the security parameter, and outputs a public key, private key pair $(e, d) \in \{0,1\}^* \times \{0,1\}^*$.

Sign is the signing algorithm. It takes as input the private key d and a message $m \in \{0,1\}^*$ and outputs a signature $\sigma \in \{0,1\}^*$.

Vrfy is the verification algorithm. It takes as input the public key e and (m, σ) and outputs "valid" or "invalid."

The digital signature scheme is correct if

$$\text{Vrfy}(e, m, \text{Sign}(d, m)) = \text{valid}$$

Note: Unlike a MAC, Vrfy cannot just generate a new signature, since that would require the private key.

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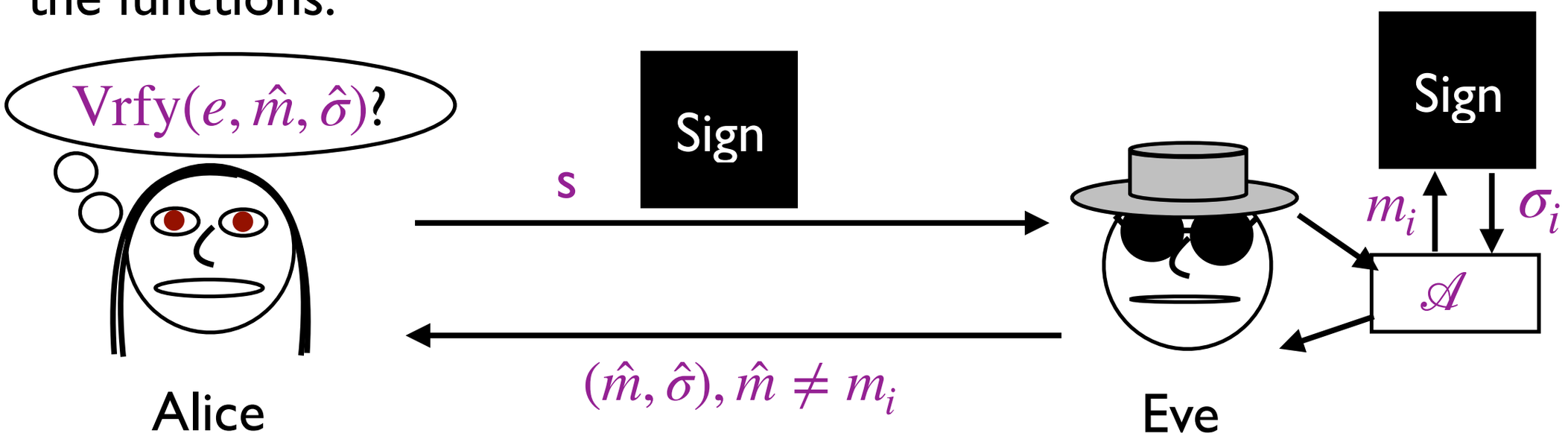
Digital Signature Security Definition

Definition: A digital signature (Gen, Sign, Vrfy) with security parameter s is secure (against an adaptive chosen-message attack) if, for any polynomial-time attack $\mathcal{A}$ with the public key e and oracle access to $\mathrm{Sign}(d, m)$, where $\mathcal{A}$ outputs $(\hat{m}, \hat{\sigma})$ such that $\mathcal{A}$ never queried the oracle for $m = \hat{m}$,

$$\Pr(\mathrm{Vrfy}(e, \hat{m}, \hat{\sigma}) = \mathrm{valid}) \leq \epsilon(s)$$

where $\epsilon(s)$ is a negligible function and the probability is averaged over (e,d) generated by Gen and the randomness used in any of the functions.



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RSA Digital Signatures

Gen: Generate two random primes p and q which are s bits long. Let N = pq. Choose $e, d \in \mathbb{Z}_N^*$ such that $ed = 1 \bmod \varphi(N)$.

The public key is (N, e) and the private key is (N, d).

Sign: Given message m and private key (N, d). The signature is $\sigma = m^d \bmod N$.

Vrfy: Given (message, signature) pair (m, σ) and public key (N, e). The message is accepted as valid if $m = \sigma^e \bmod N$.

Vote: Is this secure (yes/no/unknown)?

(If RSA assumption is true.)

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Answer: No.

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Attack on Plain RSA Signatures

What if we come up with the signature σ *first*?

> We then need to find a message m such that $\sigma = m^d \bmod N$.

How can we do this?

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> m decrypts to σ, so σ encrypts to m.

> In this case, Eve picks a random σ and therefore gets a random m. This is maybe of limited practical use, but it is definitely a violation of the security definition.

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And notice that this attack doesn't even require Eve to see any valid signatures.

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Forging a Specific Message

Suppose we want to forge a specific message m.

Notice: Given signatures for two messages m_1 and m_2,

$$\sigma_1 = m_1^d \bmod N$$

$$\sigma_2 = m_2^d \bmod N$$

Then

$$\sigma_1\sigma_2 = m_1^d m_2^d = (m_1 m_2)^d \bmod N$$

That is, the signature of the message $m_1 m_2$ is $\sigma_1\sigma_2$.

Therefore, to forge the message $m_1 m_2$, all we need are the signatures of the two messages m_1 and m_2.

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Revised RSA Signatures

New Idea: Put m through a hash function H first.

Gen: Generate two random primes p and q which are s bits long. Let N = pq. Choose $e, d \in \mathbb{Z}_N^*$ such that $ed = 1 \bmod \varphi(N)$.

The public key is (N, e) and the private key is (N, d).

Sign: Given message m and private key (N, d). The signature is $\sigma = H(m)^d \bmod N$.

Vrfy: Given (message, signature) pair (m, σ) and public key (N, e). The message is accepted as valid if $H(m) = \sigma^e \bmod N$.

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(If RSA assumption is true.)

Answer: Yes, if H is a random oracle. The standards use a padded hash function instead, in which case the answer is unknown.

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Security Discussion

With the revised definition, the first attack fails because finding m given σ requires inverting H(m).

But it is hard to invert a random oracle.

Since $H(m_1)$, $H(m_2)$, and $H(m_1 m_2)$ are random, they have no relationship to each other and in particular, $H(m_1 m_2) \neq H(m_1)H(m_2)$.

This foils the second attack.

However, we do need to be careful that we can't find multiplicative relations among the outputs of the $H(m_i)$.

Padding H helps to achieve this in practice, but there is no proof that the particular schemes which are widely used actually work — but no attacks are known despite having been used for many years.

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Same Key for Encryption & Signature

What if Bob decides to use the same N and the same (private key, public key) pair (e,d) for encryption and signatures?

> Suppose Bob is using plain RSA encryption and signatures. They are already insecure, but this makes an even more powerful attack possible:

Alice sends Bob a ciphertext $c = m^e \bmod N$. Eve intercepts it and convinces Bob to sign the "message" c. Bob then produces the signature $\sigma = c^d = m^{ed} = m \bmod N$! Bob's signature is a decryption of Alice's message, revealing the message to Eve.

> When correctly using the hash function with the signature, this particular attack doesn't work since c won't be correctly padded and it will be hard to find a message x to sign that gives H(x) = c, but this still seems like a vulnerability.

There are also good key management reasons not to do this.

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Identification Schemes

Suppose Bob has Alice's public key. Bob is talking to someone who claims to be Alice and wants proof. Alice doesn't want to reveal information that would let Bob impersonate her.

This is a "proof of knowledge."

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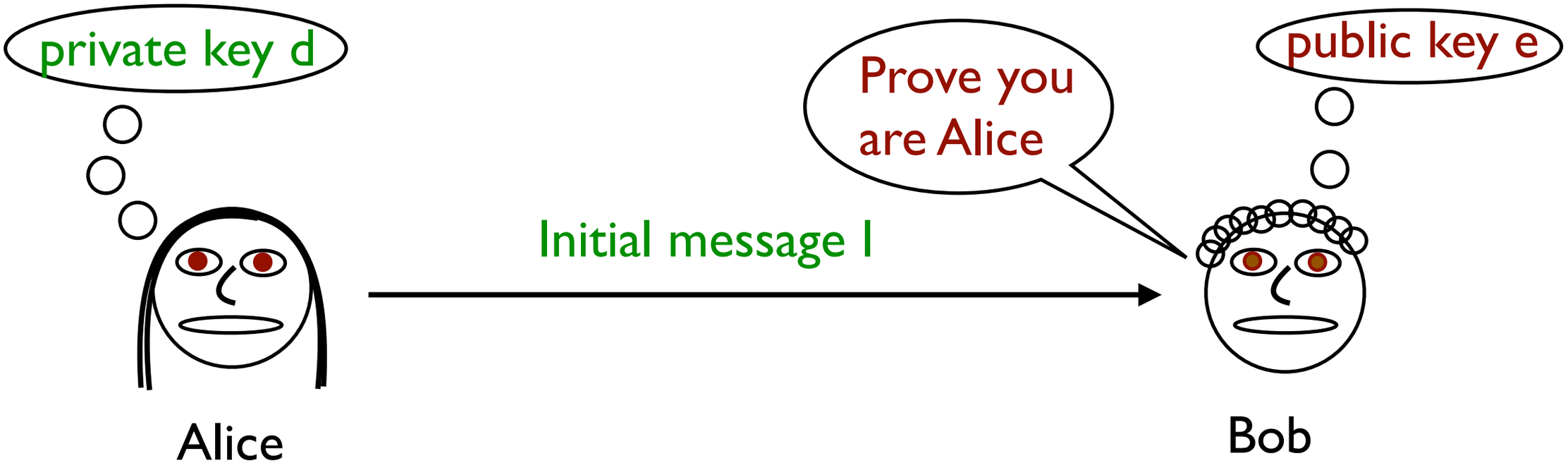

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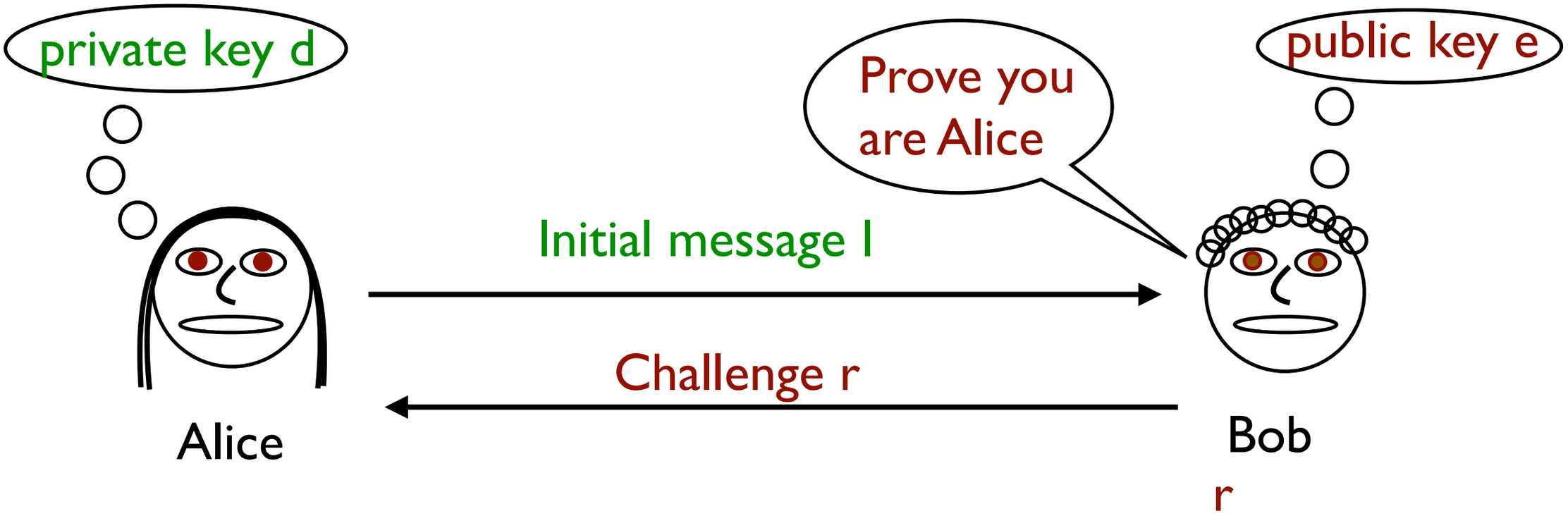

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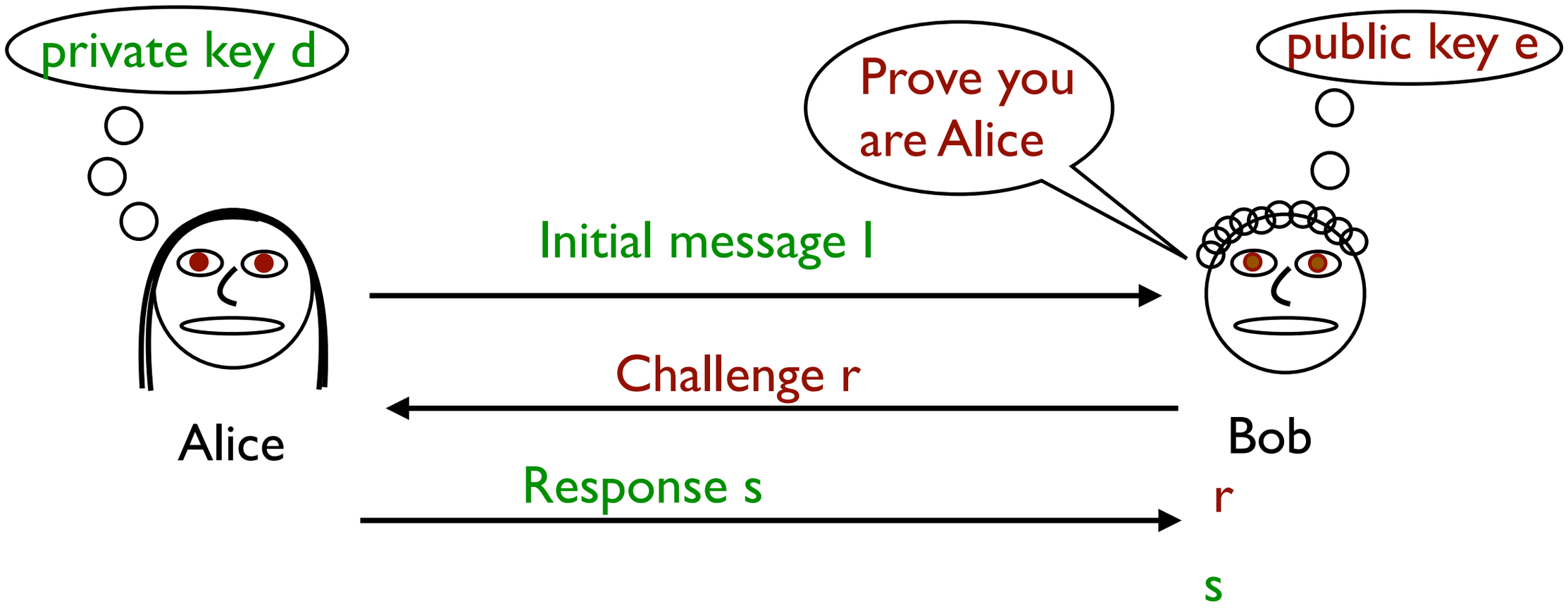

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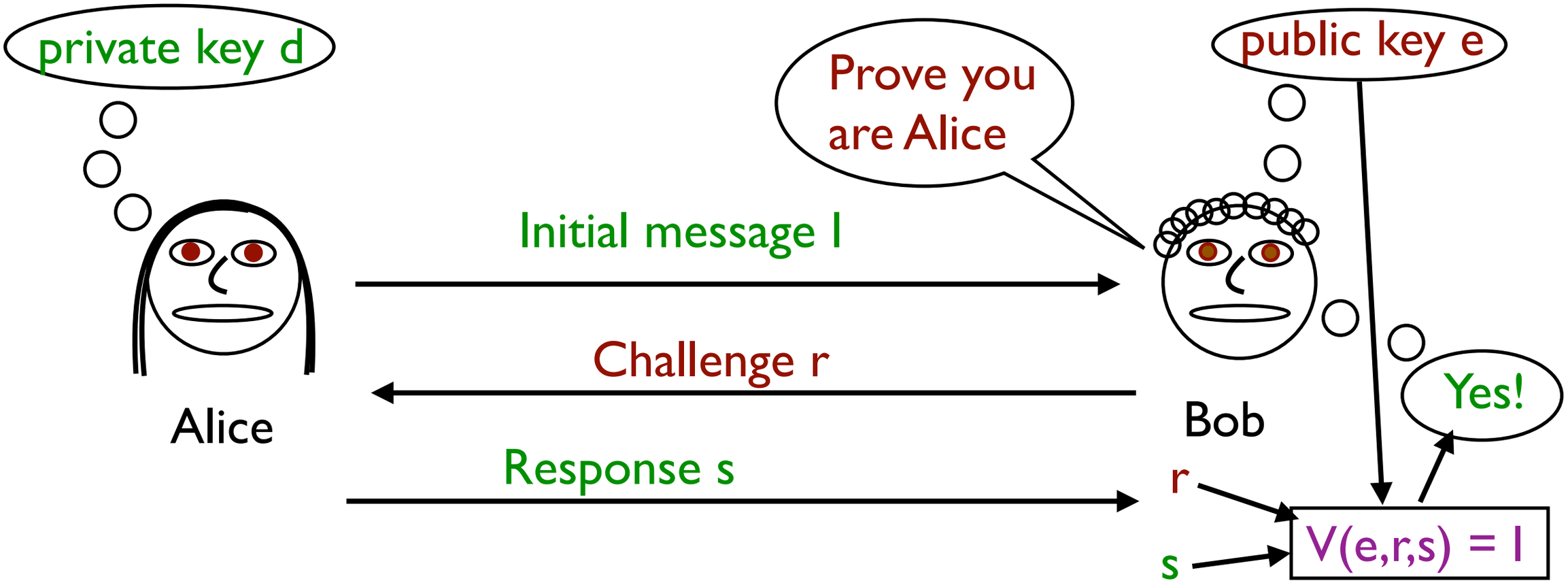

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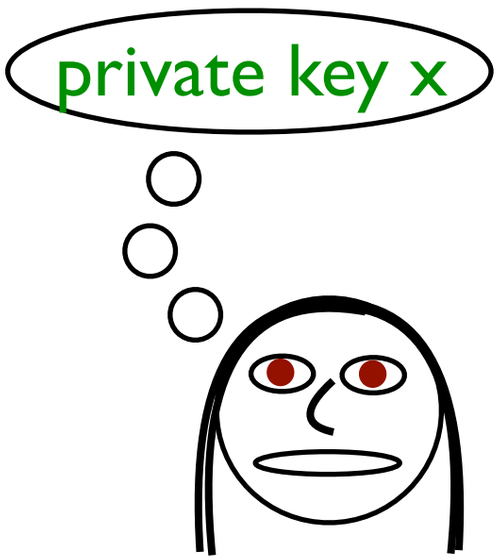
Discrete-Log Identification

Suppose Alice uses a El Gamal-like public key: Prime p with large prime order q subgroup of $\mathbb{Z}_p^*$, base g (in the order q subgroup), and y, with $y = g^x \bmod p$. Here x is the private key.



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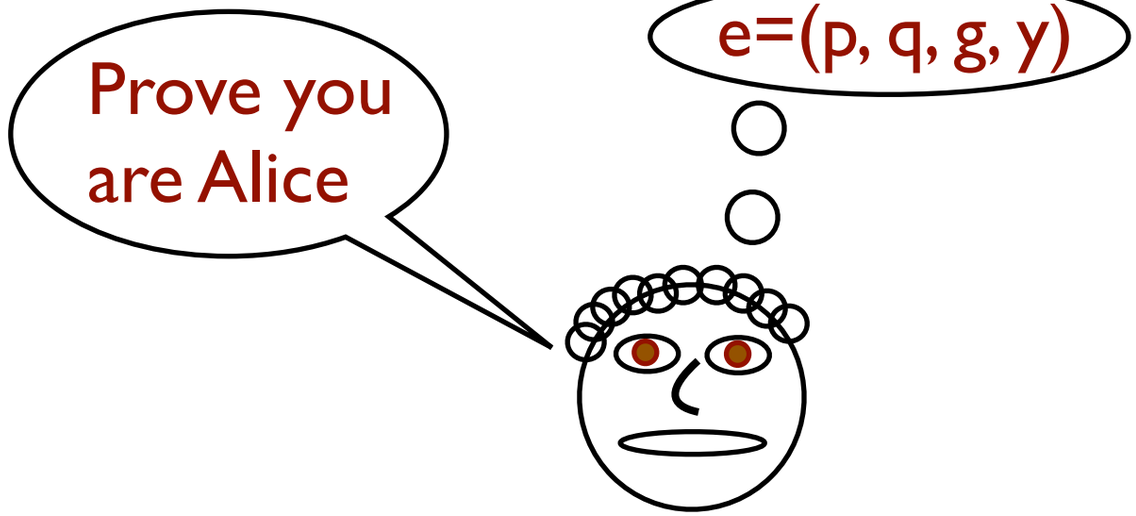

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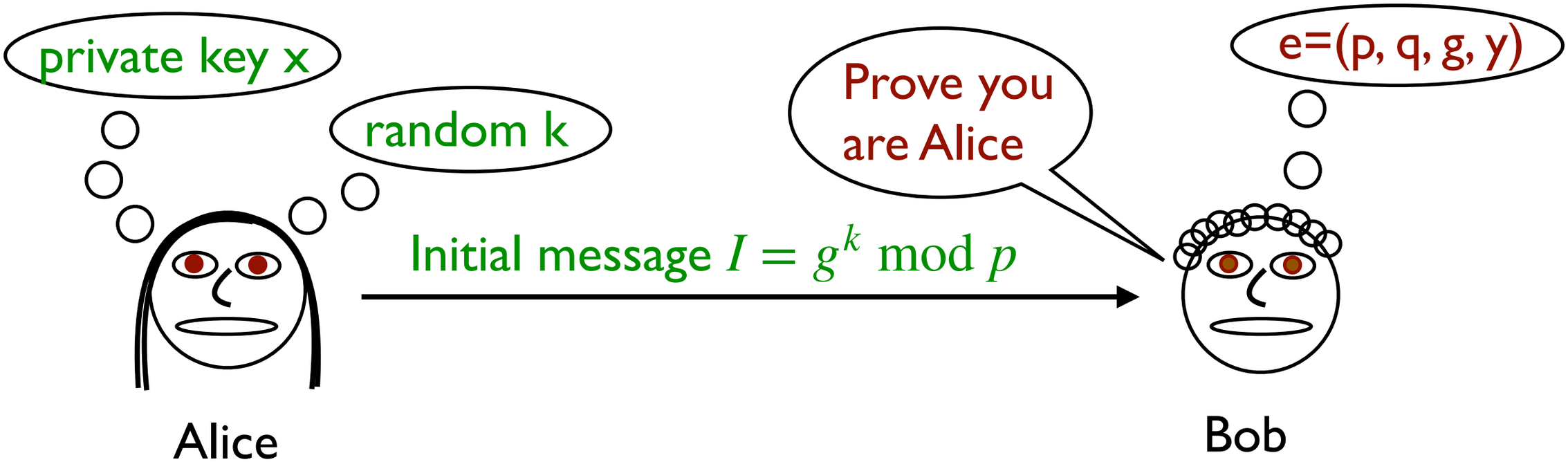

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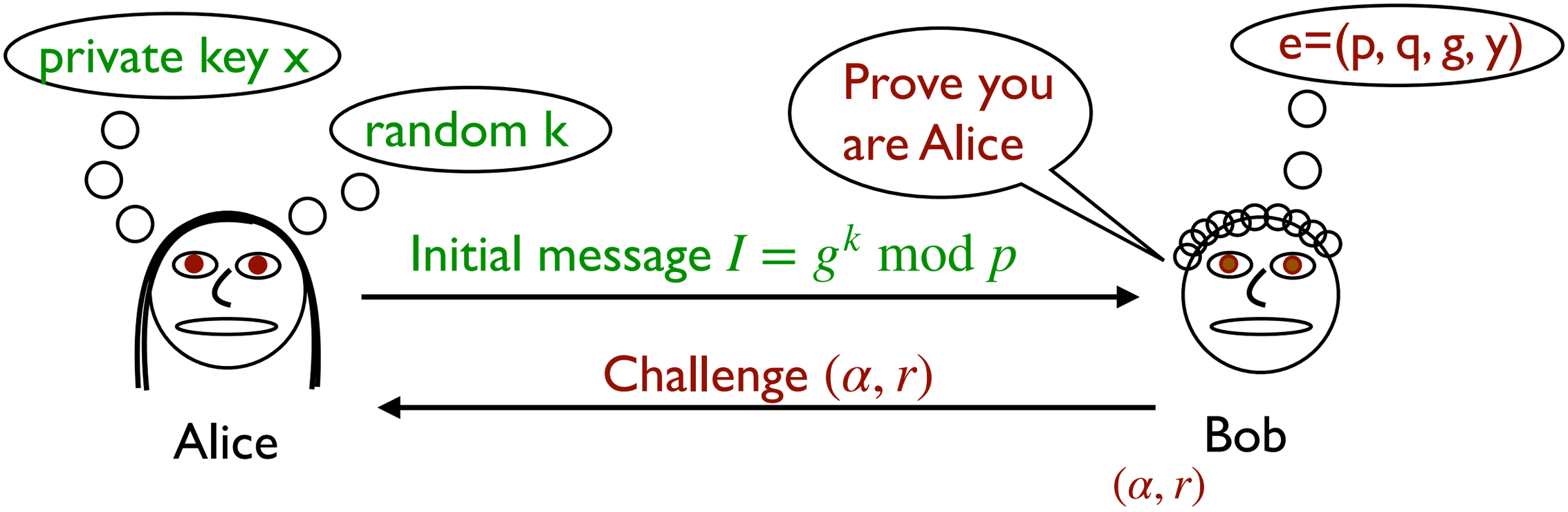

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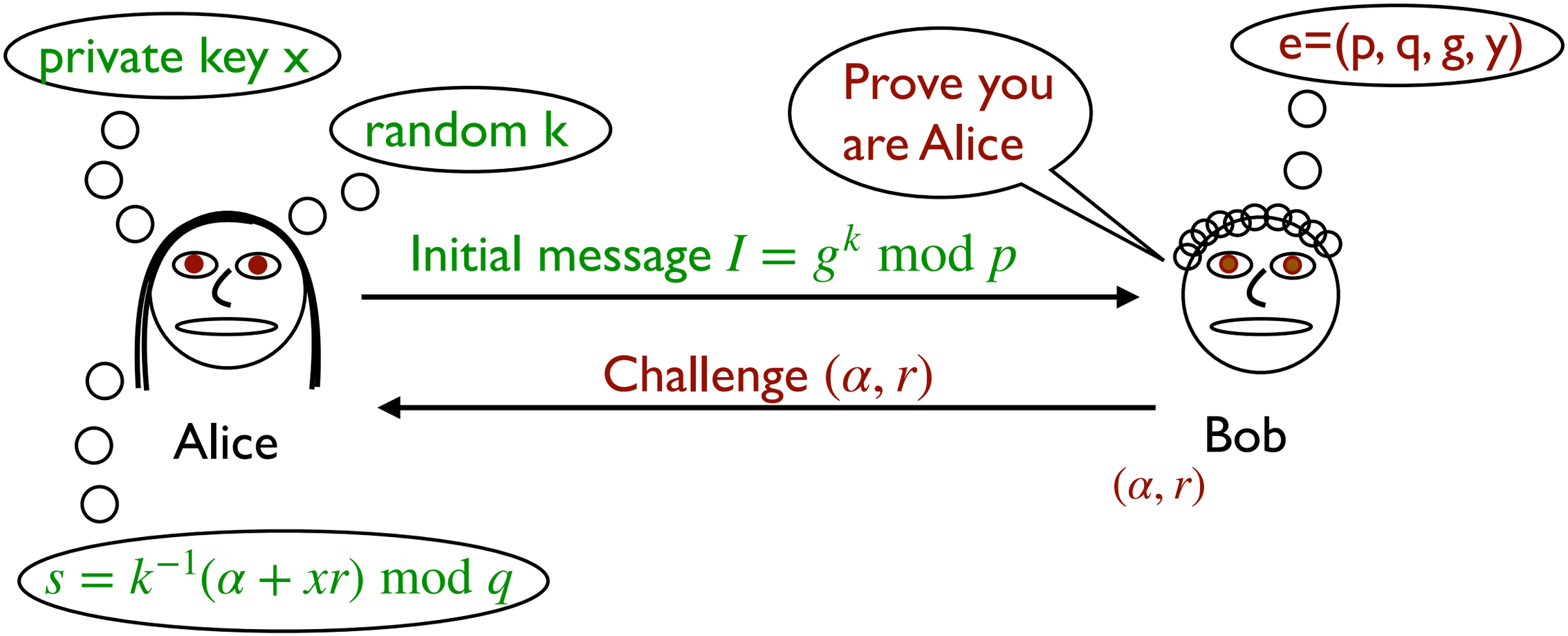

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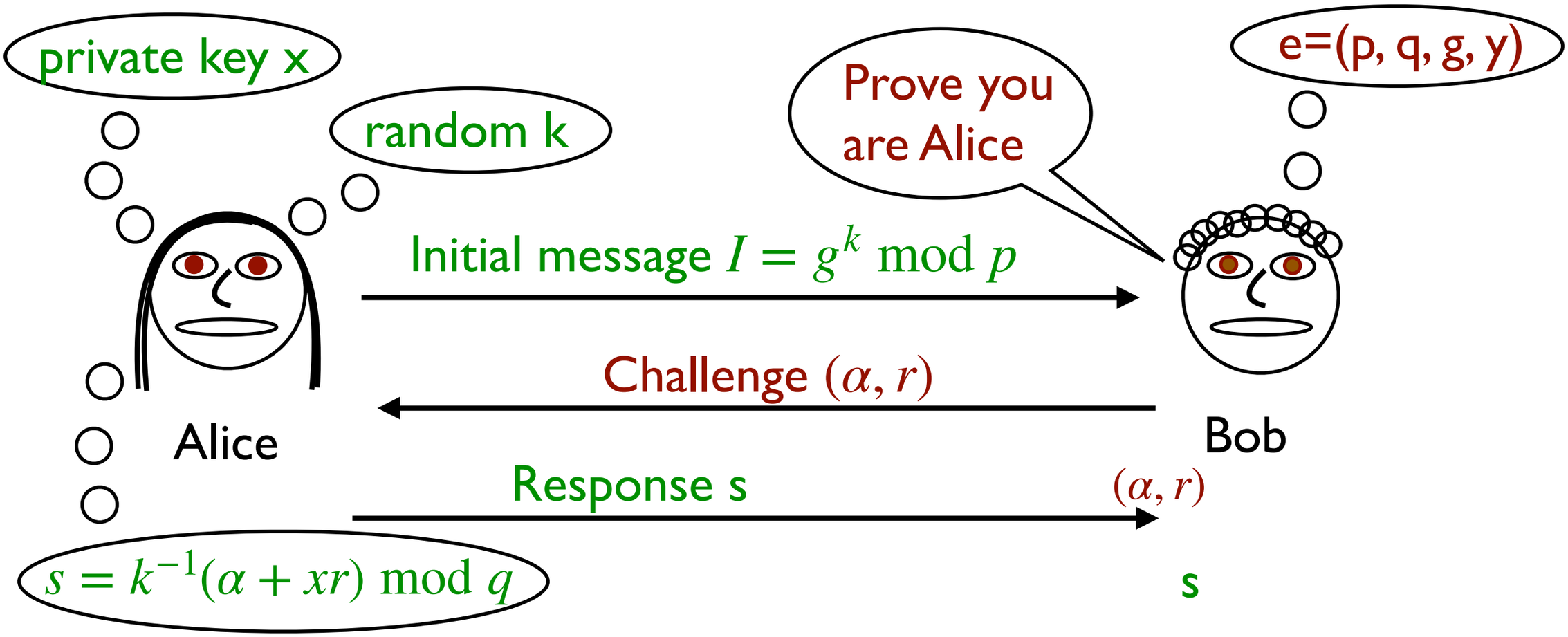

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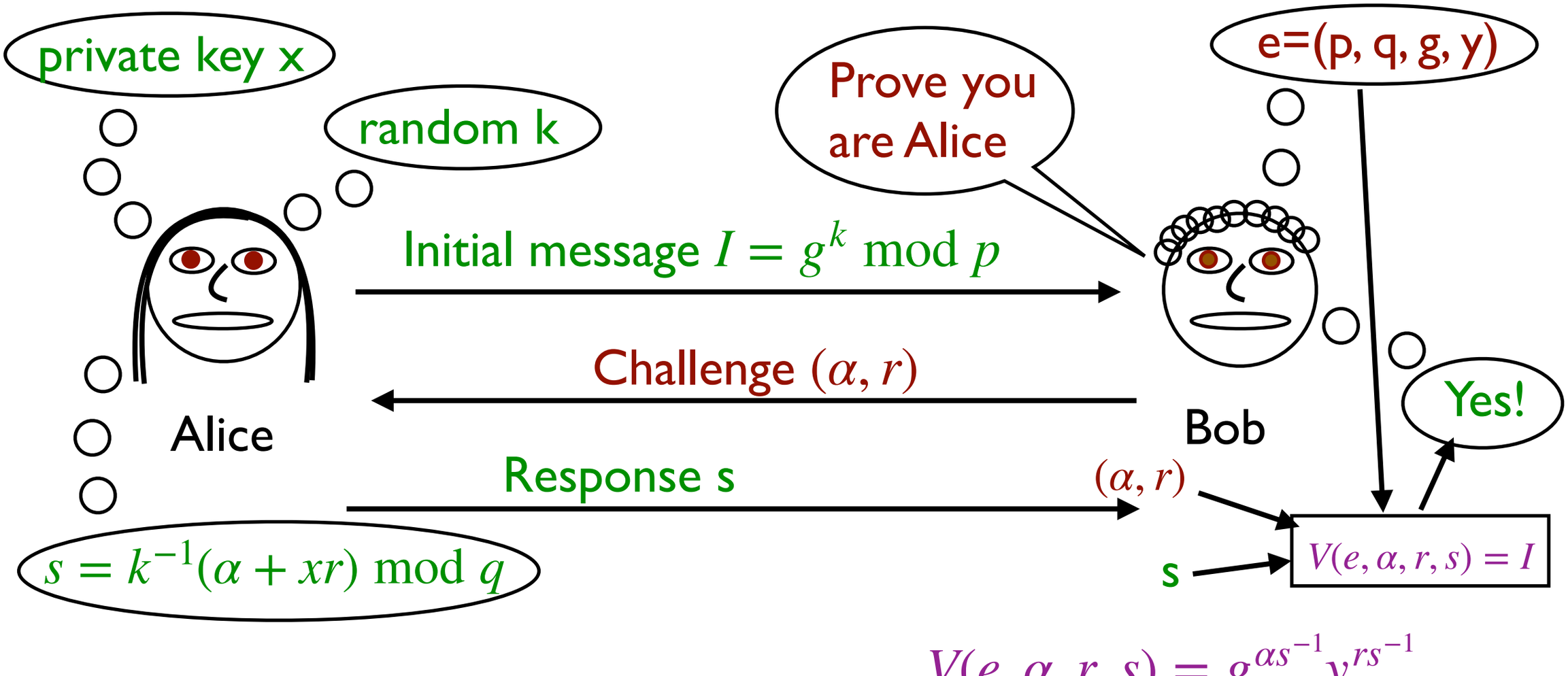

$$V(e, \alpha, r, s) = g^{\alpha s^{-1}} y^{rs^{-1}}$$

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Correctness of Identification

When Alice is actually present she can pass this test:

$$
\begin{aligned}
V(e, \alpha, r, s) &= g^{\alpha s^{-1}} y^{rs^{-1}} \bmod p \\
&= (g^{\alpha} y^{r})^{s^{-1}} \bmod p \\
&= (g^{\alpha + rx})^{s^{-1}} \bmod p \qquad \longleftarrow \quad y = g^{x} \bmod p \\
&= g^{(\alpha + rx)s^{-1}} \bmod p \\
&= g^{k} \bmod p \\
&= I
\end{aligned}
$$

$$s = k^{-1}(\alpha + xr) \bmod q$$

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Security of Identification

Roughly speaking, the identification protocol is secure if Eve can't masquerade as Alice even after seeing transcripts of previous identification sessions.

Why is this secure? It is possible to generate fake transcripts by generating them out of order: First generate random α, r, and s, and then pick $I = g^{\alpha s^{-1}} y^{rs^{-1}}$.

These fake transcripts don't use any knowledge of x, so they could have been generated by Eve and don't help her break the identification scheme.

Can solve discrete log if can give correct responses to two different (α, r) pairs for same I…

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Reduction from Discrete Log

Suppose we have an attack $\mathscr{A}$ which succeeds given random (α, r) in response to challenge I. We can run it twice using challenges (α, r_1) and (α, r_2) (but using the same internal random bits for $\mathscr{A}$ so that I is the same). Then we get s_1 and s_2 such that

$$g^{\alpha s_1^{-1}} y^{r_1 s_1^{-1}} = I = g^{\alpha s_2^{-1}} y^{r_2 s_2^{-1}} \bmod p$$

Thus,

$$y^{r_1 s_1^{-1} - r_2 s_2^{-1}} = g^{\alpha(s_1^{-1} - s_2^{-1})} \bmod p$$

Since g and y have order q, if we let

$$x = \alpha(s_1^{-1} - s_2^{-1})(r_1 s_1^{-1} - r_2 s_2^{-1})^{-1} \bmod q$$

then

$$y = g^x \bmod p$$

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DSA (Discrete-Log Signatures)

Gen: The primes p and q and base g have been pre-determined, along with hash function H. Gen chooses a random $x \in \{0,\ldots,\mathrm{ord}(g)-1\}$, which becomes the private key. I.e., d = x. The public key is $y = g^x \bmod p$.

Sign: Given message m and private key x. Choose random $k \in \{0,\ldots,\mathrm{ord}(g)-1\}$ and let $r = g^k \bmod p$. The signature is $\sigma = (r,s)$, where $s = k^{-1}(H(m)+xr) \bmod q$.

Vrfy: Given m, σ, and y. Vrfy checks if $r = g^{H(m)s^{-1}} y^{rs^{-1}} \bmod p$.

Correctness: Same as identification scheme.

Security: Unproven, even when H is a random oracle.

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Repeated k

k must be new each time and kept hidden from Eve.

If k is repeated:

Then r repeats and (m_1, s_1) and (m_2, s_2) are known with:

$$s_1 = k^{-1}(H(m_1) + xr) \bmod q$$

$$s_2 = k^{-1}(H(m_2) + xr) \bmod q$$

$$s_1 - s_2 = k^{-1}(H(m_1) - H(m_2)) \bmod q$$

$$k = (s_1 - s_2)^{-1}(H(m_1) - H(m_2)) \bmod q$$

Once k is known:

$$s = k^{-1}(H(m) + xr) \bmod q$$

$$x = r^{-1}(ks - H(m)) \bmod q$$

and Eve learns the private key x.

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