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Check the latex file and find out how macros are used to display basic quantum objects!

Quantum States. The state space $\mathcal{A}$ of m-qubit is the complex Euclidean space $\mathbb{C}^{2^m}$. An m-qubit quantum state is represented by a density operator ρ, i.e., a positive semidefinite operator over $\mathcal{A}$ with trace 1. The set of all quantum states in $\mathcal{A}$ is denoted by $\mathrm{Dens}\,(\mathcal{A})$.

Let $\mathrm{L}\,(\mathcal{A})$ denote the set of all linear operators on space $\mathcal{A}$. The Hilbert-Schmidt inner product on $\mathrm{L}\,(\mathcal{A})$ is defined by $\langle X, Y\rangle = \mathrm{tr}(X^*Y)$, for all $X, Y \in \mathrm{L}\,(\mathcal{A})$, where X^* is the adjoint conjugate of X. Let $\mathsf{id}_{\mathcal{X}}$ denote the identity operator over $\mathcal{X}$, which might be omitted from the subscript if it is clear in the context. An operator $U \in \mathrm{L}\,(\mathcal{X})$ is a unitary if $UU^* = U^*U = \mathsf{id}_{\mathcal{X}}$. The set unitary operations over $\mathcal{X}$ is denoted by $U(\mathcal{X})$.

For a multi-partite state, e.g. $\rho_{ABC} \in \mathrm{Dens}\,(\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{C})$, its reduced state on some subsystem(s) is represented by the same state with the corresponding subscript(s). For example, the reduced state on $\mathcal{A}$ system of ρ_{ABC} is $\rho_A = \mathrm{tr}_{\mathcal{BC}}(\rho_{ABC})$, and $\rho_{AB} = \mathrm{tr}_{\mathcal{C}}(\rho_{ABC})$. When all subscript letters are omitted, the notation represents the original state (e.g., $\rho = \rho_{ABE}$).

A *classical-quantum-*, or cq-state $\rho \in \mathrm{Dens}\,(\mathcal{A}\otimes\mathcal{B})$ indicates that the $\mathcal{A}$ subsystem is classical and $\mathcal{B}$ is quantum. Likewise for ccq-, etc., states. We use *lower case* letters to denote specific values assignment to the classical part of a state. For example, any cq-state $\rho_{AB} = \sum_a p_a \,|a\rangle\langle a| \otimes \rho_B^a$ in which $p_a = \mathbf{Pr}[A = a]$ and ρ_B^a is a normalized state.

Distance Measures. For any $X \in \mathrm{L}\,(\mathcal{A})$ with singular values $\sigma_1, \cdots, \sigma_d$, where $d = \dim(\mathcal{A})$, the trace norm of $\mathcal{A}$ is $\|X\|_{\mathrm{tr}} = \sum_{i=1}^d \sigma_i$.

The *trace distance* between two quantum states ρ_0 and ρ_1 is defined to be

$$|\rho_0 - \rho_1|_{\mathrm{tr}} \stackrel{\mathrm{def}}{=} \frac{1}{2}\|\rho_0 - \rho_1\|_{\mathrm{tr}}\,.$$

, which admits the following operational meaning. The following well-known fact relates the trace distance with the optimal probability of distinguishing quantum states. Their *fidelity*, denoted by $\mathrm{F}(\rho_0, \rho_1)$), is

$$\mathrm{F}(\rho_0, \rho_1) = \|\sqrt{\rho_0}\sqrt{\rho_1}\|_{\mathrm{tr}}\,. \tag{1}$$

When ρ_0 and ρ_1 are *classical* states, the trace distance $|\rho_0 - \rho_1|_{\mathrm{tr}}$ is equivalent to the *statistical* distance between ρ_0 and ρ_1. It is also a well known fact that for two distributions X_1, X_2 over $\mathcal{X}$, let $p_x = \mathbf{Pr}[X_1 = x]$ and $q_x = \mathbf{Pr}[X_2 = x]$ and their statistical distance satisfies

$$|X_1 - X_2|_{\mathrm{tr}} = \frac{1}{2}\sum_x |p_x - q_x| = \sum_{x:p_x>q_x}(p_x - q_x). \tag{2}$$

For simplicity, when both states are classical, we use $(X_1) \approx_\epsilon (X_2)$ to denote $|X_1 - X_2|_{\mathrm{tr}} \le \epsilon$.

Moreover, the trace distance admits the following two simple facts.

Fact 1. *For any state $\rho_1, \rho_2 \in \mathrm{Dens}\,(\mathcal{A})$ and $\sigma \in \mathrm{Dens}\,(\mathcal{B})$, we have*

$$|\rho_1 - \rho_2|_{\mathrm{tr}} = |\rho_1\otimes\sigma - \rho_2\otimes\sigma|_{\mathrm{tr}}\,.$$

Fact 2. *Let $\rho, \sigma \in \mathrm{Dens}\,(\mathcal{A}\otimes\mathcal{B})$ be any two cq-states where $\mathcal{A}$ is the classical part. Moreover, $\rho = \sum_a p_a\,|a\rangle\langle a|\otimes\rho_B^a$ and $\sigma = \sum_a q_a\,|a\rangle\langle a|\otimes\sigma_B^a$. Then we have*

$$|\rho - \sigma|_{\mathrm{tr}} = \sum_a |p_a\rho_B^a - q_a\sigma_B^a|_{\mathrm{tr}}\,.$$

References

[1] Donald E. Knuth (1986) *The TEX Book*, Addison-Wesley Professional.

[2] Leslie Lamport (1994) *LATEX: a document preparation system*, Addison Wesley, Massachusetts, 2nd ed.