CMSC/Math 456: Cryptography (Fall 2022)

Lecture 10

Daniel Gottesman

Administrative

Problem set #4 should have been turned in. Solutions for problem set #3 are on ELMS.

Problem set #5 is available now, due next Thursday, Oct. 7.

This class is being recorded

$\mathbb{Z}_N$ and $\mathbb{Z}_N^*$

We have been working numbers modulo N. I will start using the notation $\mathbb{Z}_N$ to refer to this.

But, as we have seen, nice things happen if we restrict attention to g that is relatively prime to N, $\gcd(g, N) = 1$:

- Division is well-defined for such g
- $\exists r$ (="there exists r") such that $g^r = 1 \bmod N$. That is, exponentiation cycles back to 1.

Definition: Let $\mathbb{Z}_N^* \subseteq \mathbb{Z}_N$ be the set of $g \in \{0, \ldots, N-1\}$ such that $\gcd(g, N) = 1$.

Closure of $\mathbb{Z}_N^*$

Proposition: If $\gcd(g, N) = 1$ and $\gcd(h, N) = 1$, then $\gcd(gh, N) = 1$ as well. I.e., $\mathbb{Z}_N^*$ is closed under multiplication.

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This means that gh has an inverse and therefore $\gcd(gh, N) = 1$.

Groups

Definition: A group (G, *) is a set G of elements along with a binary operation $*: G \times G \to G$ with the following properties:

1. Closure: $g * h \in G$ when $g, h \in G$.
2. Associativity: $\forall g, h, k \in G, (g * h) * k = g * (h * k)$.
3. Identity: $\exists e \in G$ such that $\forall g \in G, e * g = g * e = g$.
4. Inverses: $\forall g \in G, \exists g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.

A group which also satisfies

5. Commutativity: $\forall g, h \in G, g * h = h * g$

is called an abelian group.

Usually we just refer to G as the group. If we need to specify the group operation, we say "G under [operation]." Usually instead of *, the group operation is just written + or $\cdot$ like addition or multiplication even if it is not those.

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Group Examples

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Integers $\mathbb{Z}$? Vote.

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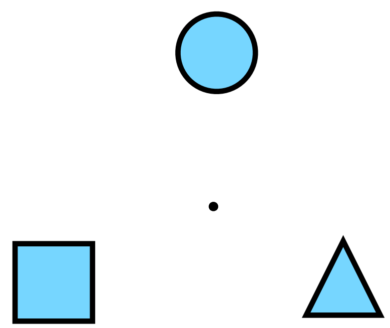
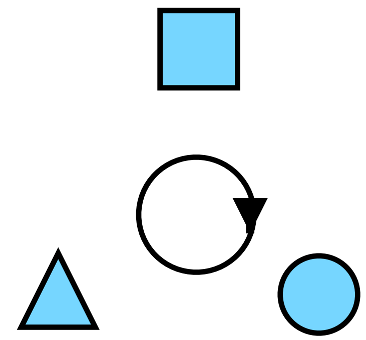
Group Theory Example

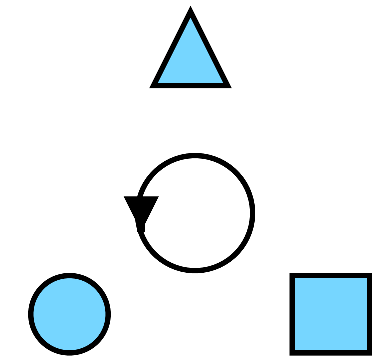
Permutations of 3 elements: Group S_3, $|S_3| = 6$



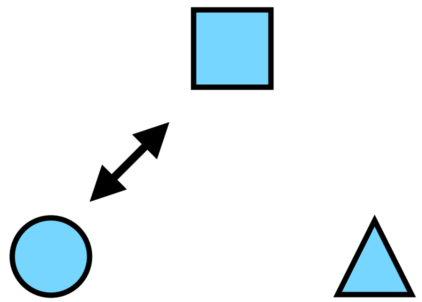
Identity e



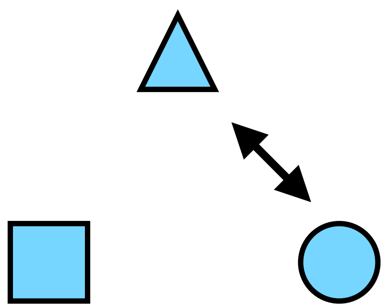
Rotate clockwise R



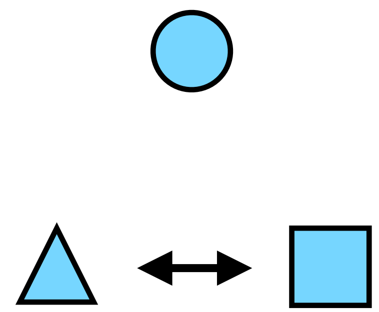
Rotate ccw $R^{-1} = R^2$



Swap left S_L



Swap right S_R



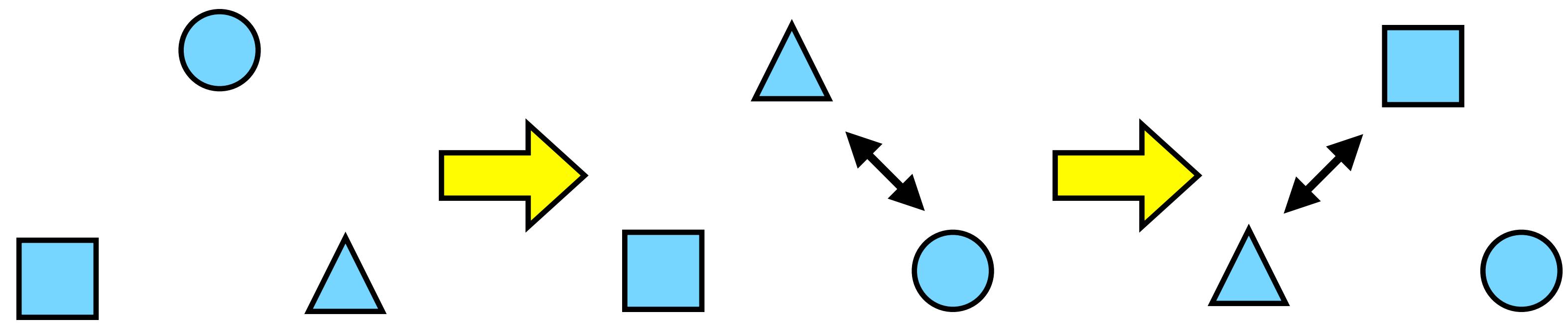
Swap bottom S_B

$S_3 = \{e, R, R^2, S_L, S_R, S_B\}$ with group operation composition.

Group Properties: Closure

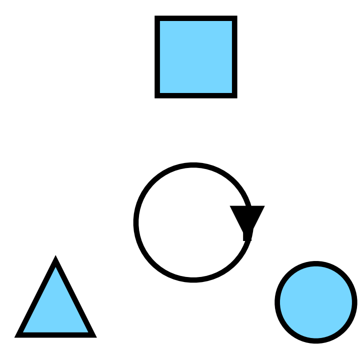
Closure: Product of two permutations is a permutation.

E.g.: $S_L S_R$ (acts from right)



Starting arrangement $\quad S_R \quad S_L$

=



$= R$

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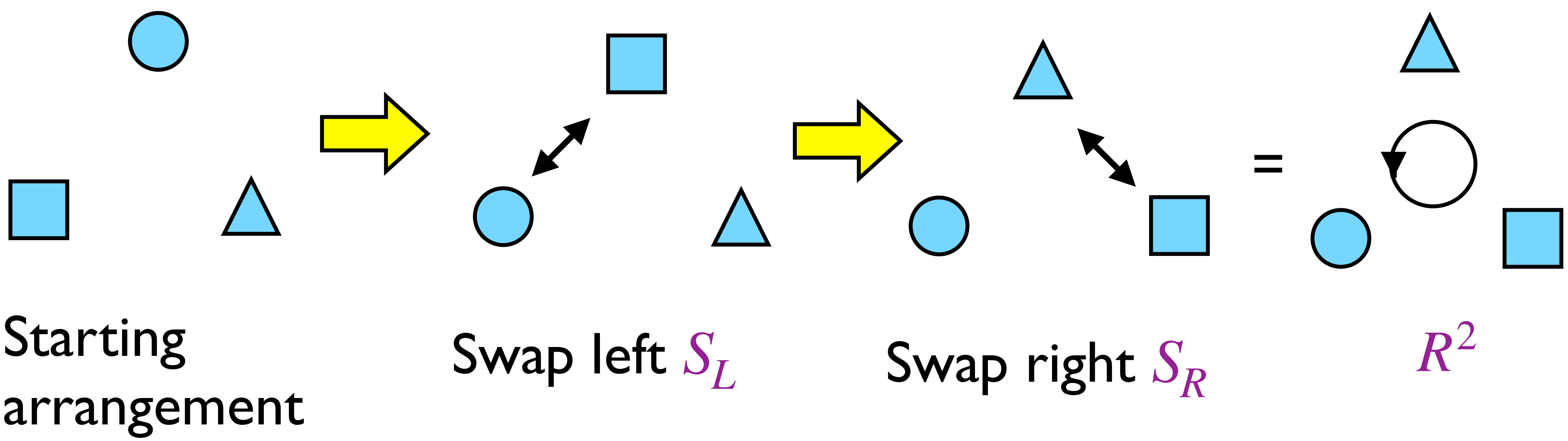
Group Properties: Others

Associativity: Can be checked but is automatic for composition of operations.

Identity: e is the identity element of the group. $e\sigma = \sigma$

Inverses: R and R^2 are inverses of each other. S_L, S_R, and S_B are inverses of themselves.

Non-abelian: $S_R S_L = R^2 \neq R = S_L S_R$



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$\mathbb{Z}_6^*$ under multiplication is not a subgroup of $\mathbb{Z}_6$ under addition because it uses a different group operation.

Order of Groups and Subgroups

Definition: The order of a finite group G is written $|G|$ and is equal to the number of elements in G.

Examples:

$|\mathbb{Z}_5| = 5$ and $|\mathbb{Z}_5^*| = 4$.

$|\mathbb{Z}_6| = 6$

$|\{0,2,4\}| = 3$

$|\mathbb{Z}_6^*| = 2$ since $\mathbb{Z}_6^* = \{1,5\}$

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G is partitioned into cosets of size $|H|$, so $|H|$ divides $|G|$.

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This contradicts $g' \neq gk$ for $k \in H$. It must therefore be that $gH \cap g'H = \emptyset$.

Proof of Claim 3

Claim 3: $|gH| = |H|$

Proof of claim 3:

$gh = gh'$ iff $h = h'$ (multiply by g^{-1}).

Thus, $gH = \{gh | h \in H\}$ has exactly as many elements as H.

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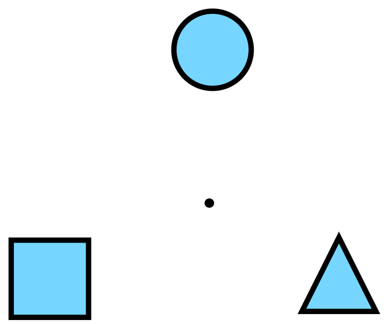
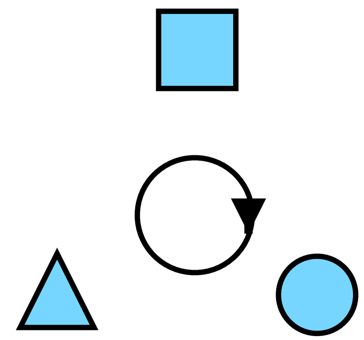
Subgroups of Permutation Group

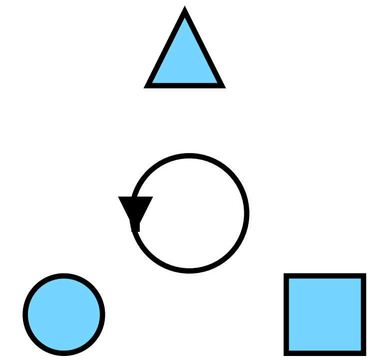
Order 3: Generated by R or by R^2.



Identity e



Rotate clockwise R



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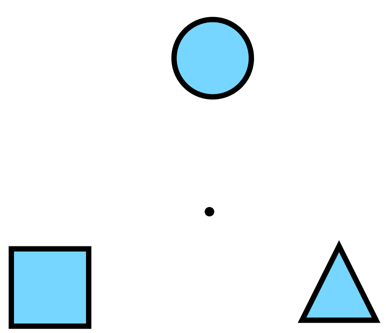
3 order 2 subgroups: For instance, generated by S_L.



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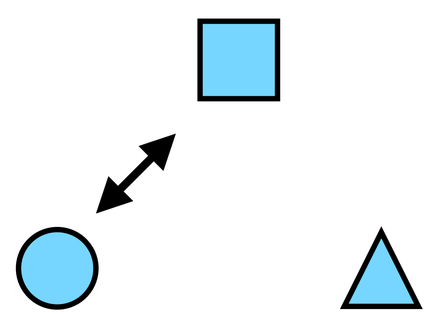

Swap left S_L

Note: Order of subgroups a factor of 6 (Lagrange’s thm.)

Generators and Cyclic Groups

Definition: Let G be a group. A set $S \subseteq G$ is a generating set for G if any element of G can be written as a finite product (under the group operation) of elements of S or inverses of elements of S, with repeats allowed. Note: S is a subset of G. It need not be a subgroup of G.

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$\{1\}$ is a generating set for $\mathbb{Z}_5$ (under addition), as is $\{a\}$ for any $a \neq 0$.

This class is being recorded

Cyclic Subgroups of $\mathbb{Z}_N^*$

Now let us consider the question: what are the possible orders of a number under modular exponentiation?

Let $g \in \mathbb{Z}_N^*$ and define $\langle g \rangle = \{g^a \in \mathbb{Z}_N^*\}$. $\langle g \rangle$ is the cyclic subgroup of $\mathbb{Z}_N^*$ generated by g.

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