Parallel Algorithms

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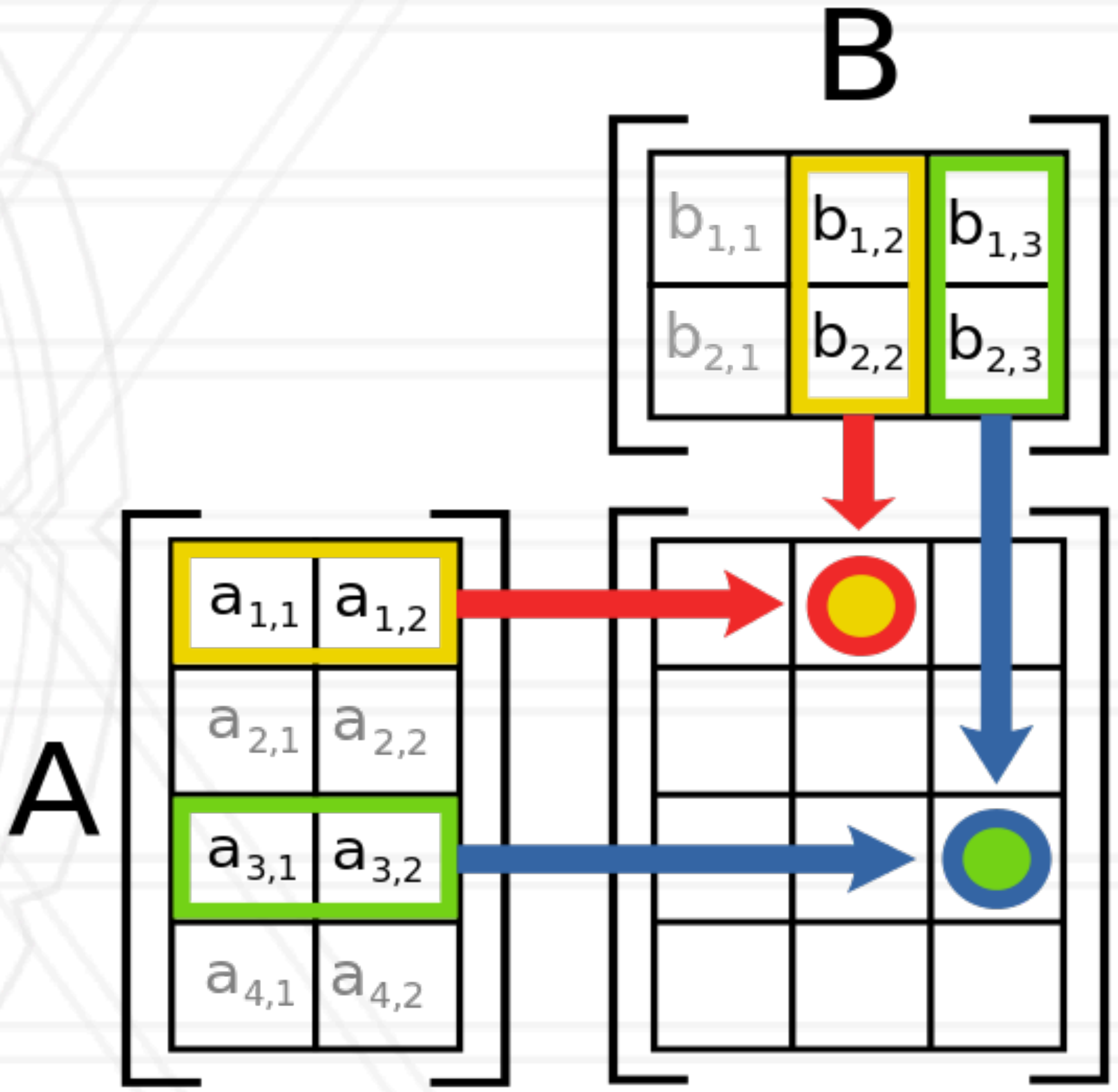
Matrix multiplication

```
for (i=0; i<M; i++)
  for (j=0; j<N; j++)
    for (k=0; k<L; k++)
      C[i][j] += A[i][k]*B[k][j];
```



https://en.wikipedia.org/wiki/Matrix_multiplication

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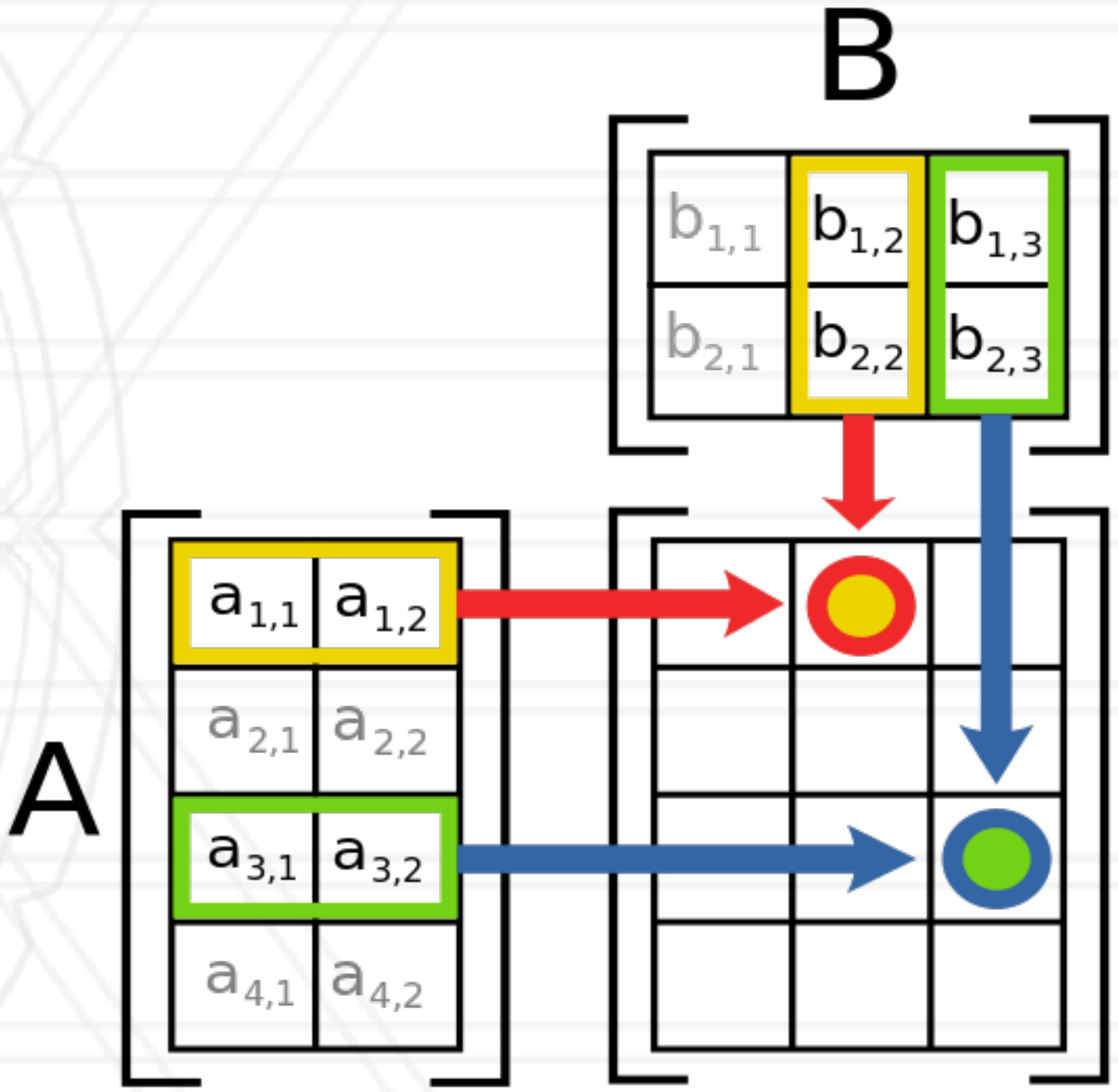
Any performance issues for large arrays?



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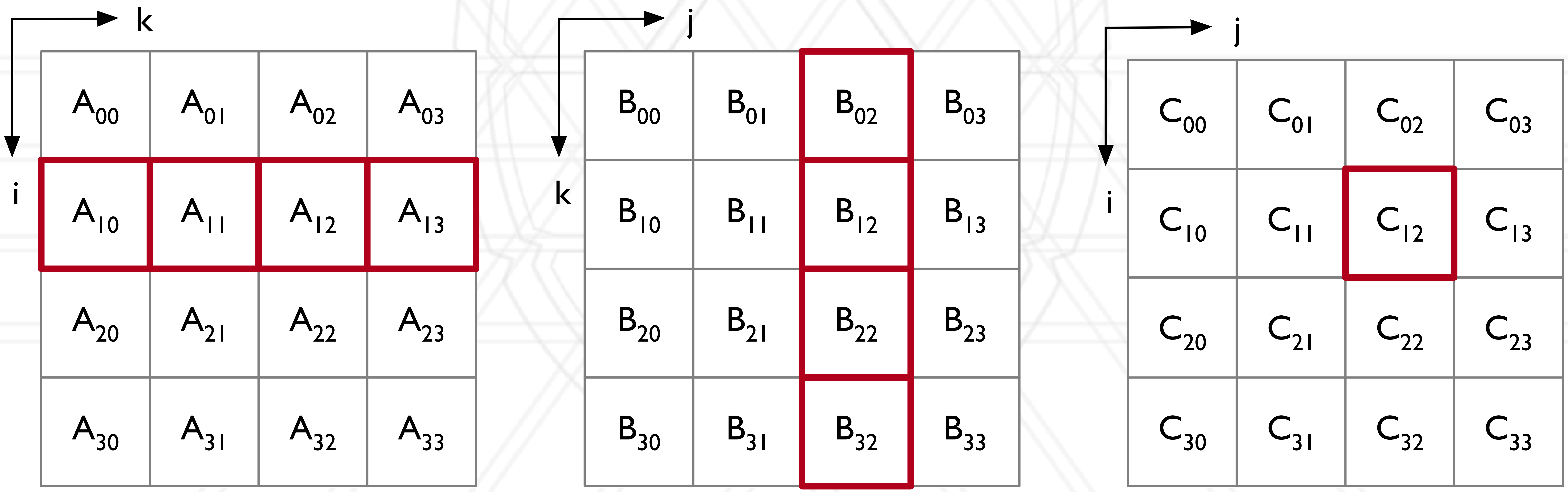
Blocking to improve cache performance

- Create smaller blocks that fit in cache: leads to cache reuse
- $C_{12} = A_{10} * B_{02} + A_{11} * B_{12} + A_{12} * B_{22} + A_{13} * B_{32}$



https://link.springer.com/chapter/10.1007/978-3-319-67630-2_36

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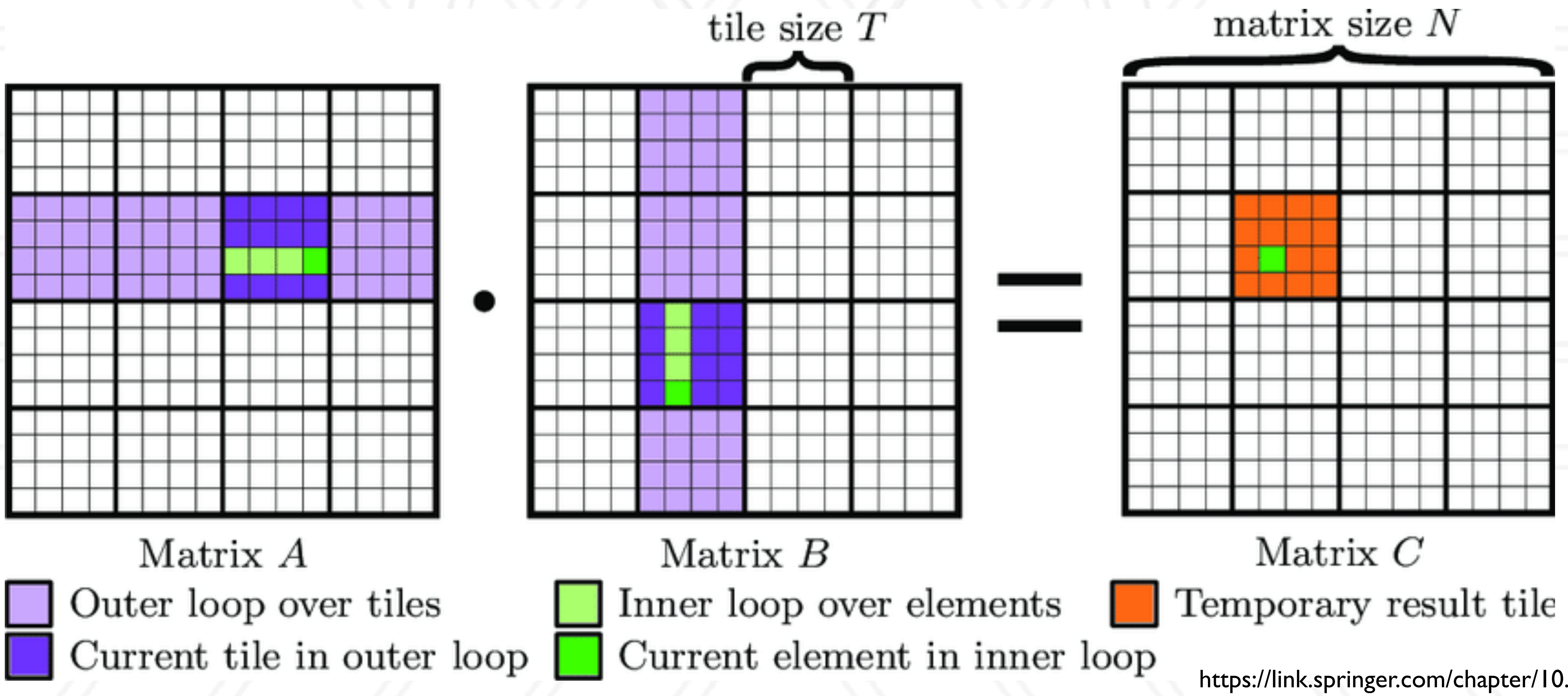

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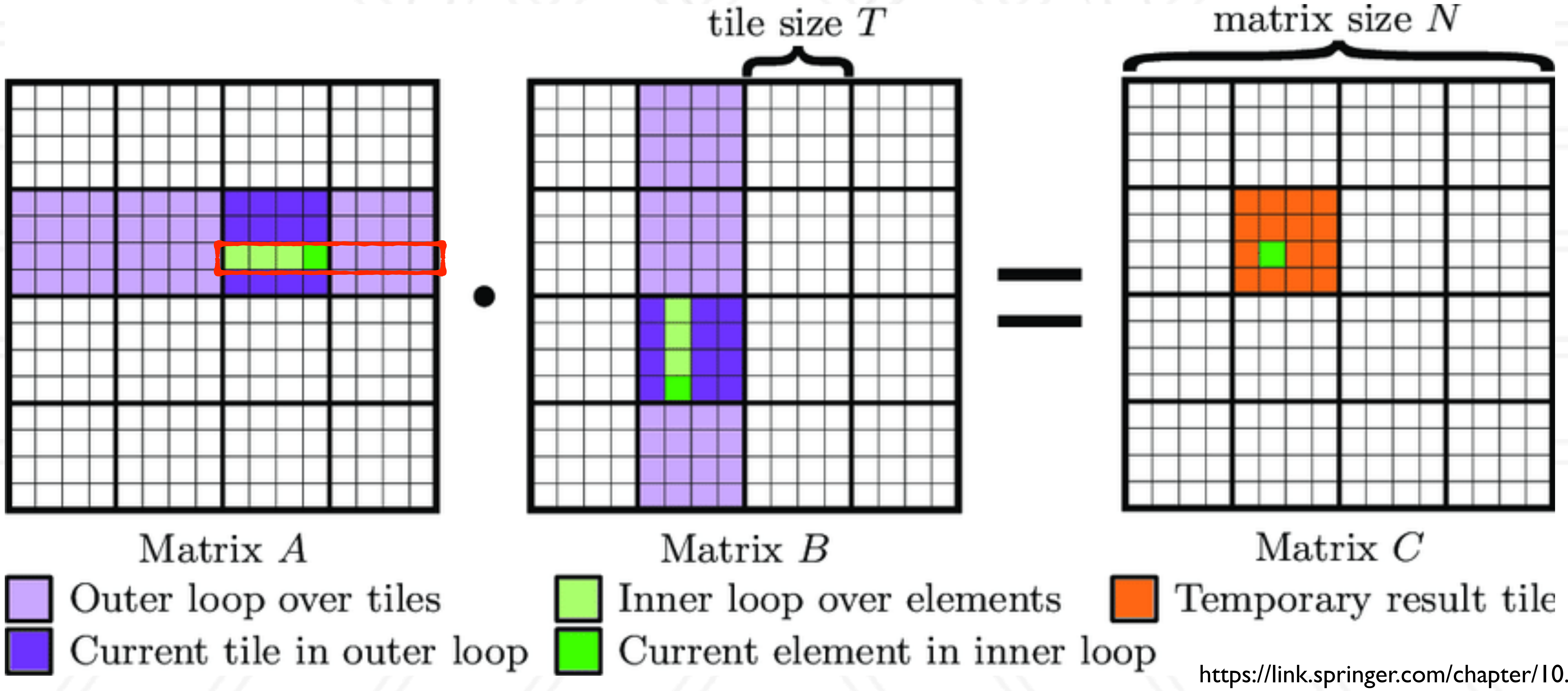

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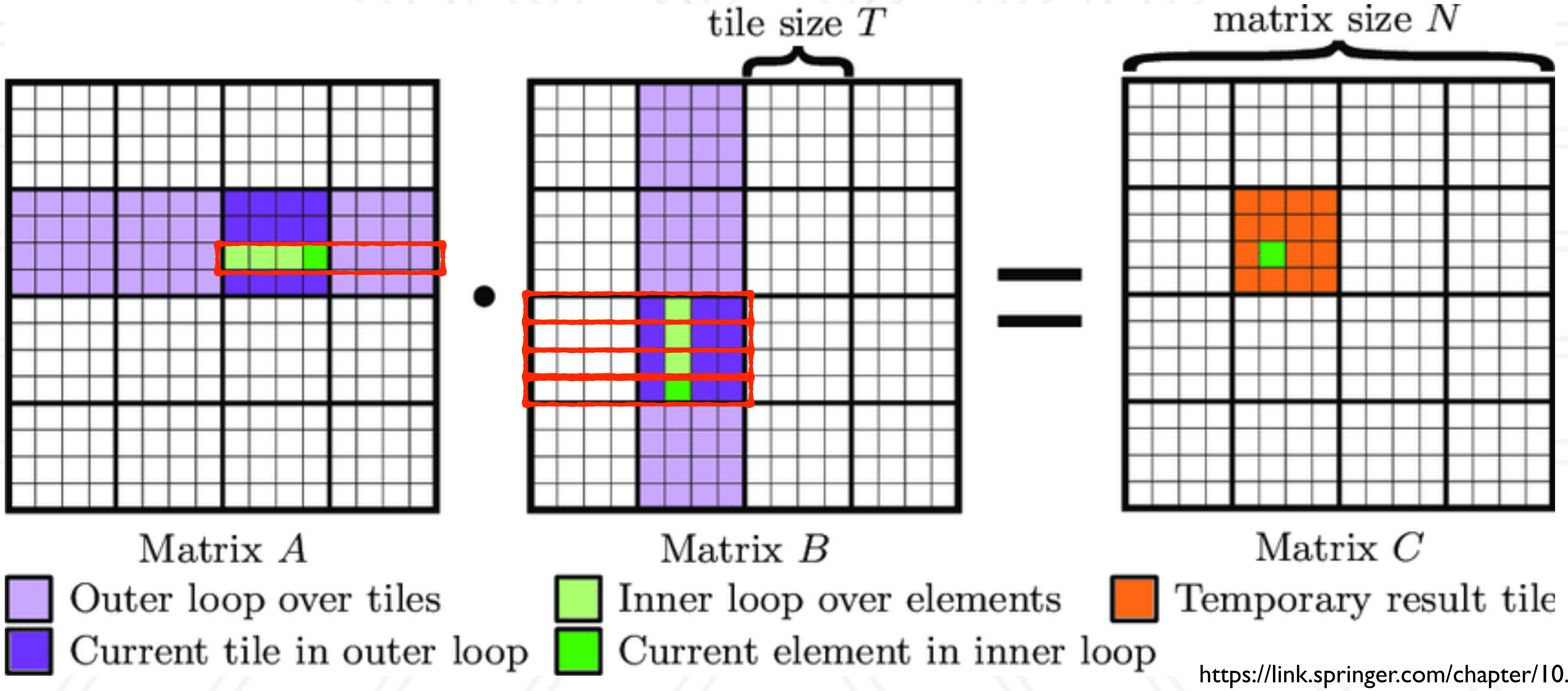

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Blocked (tiled) matrix multiply

```
for (ii = 0; ii < n; ii+=B) {
  for (jj = 0; jj < n; jj+=B) {
    for (kk = 0; kk < n; kk+=B) {
      for (i = ii; i < ii+B; i++) {
        for (j = jj; j < jj+B; j++) {
          for (k = kk; k < kk+B; k++) {
            C[i][j] += A[i][k]*B[k][j];
          }
        }
      }
    }
  }
}
```

Original code

```
for (i=0; i<M; i++)
  for (j=0; j<N; j++)
    for (k=0; k<L; k++)
      C[i][j] += A[i][k]*B[k][j];
```

Parallel matrix multiply

- Store A and B in a distributed manner
- Communication between processes to get the right sub-matrices to each process
- Each process computes a portion of C

Cannon’s 2D matrix multiply

- Arrange processes in a 2D virtual grid
- Assign sub-blocks of A and B to each process
- Each process responsible for computing a sub-block of C
- Requires other processes in its row and column to send A and B blocks so can it can compute the final values of its sub-block

Cannon's 2D matrix multiply

- $C_{12} = A_{10} * B_{02} + A_{11} * B_{12} + A_{12} * B_{22} + A_{13} * B_{32}$

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15

2D process grid

A_{00}	A_{01}	A_{02}	A_{03}
A_{10}	A_{11}	A_{12}	A_{13}
A_{20}	A_{21}	A_{22}	A_{23}
A_{30}	A_{31}	A_{32}	A_{33}

B_{00}	B_{01}	B_{02}	B_{03}
B_{10}	B_{11}	B_{12}	B_{13}
B_{20}	B_{21}	B_{22}	B_{23}
B_{30}	B_{31}	B_{32}	B_{33}

Cannon's 2D matrix multiply

- $C_{12} = A_{10} * B_{02} + A_{11} * B_{12} + A_{12} * B_{22} + A_{13} * B_{32}$

A: Displace blocks in row i by i

B: Displace blocks in column j by j

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4	5	6	7
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2D process grid

A_{00}	A_{01}	A_{02}	A_{03}
A_{10}	A_{11}	A_{12}	A_{13}
A_{20}	A_{21}	A_{22}	A_{23}
A_{30}	A_{31}	A_{32}	A_{33}

Initial skew in rows

B_{00}	B_{01}	B_{02}	B_{03}
B_{10}	B_{11}	B_{12}	B_{13}
B_{20}	B_{21}	B_{22}	B_{23}
B_{30}	B_{31}	B_{32}	B_{33}

Initial skew in columns

Cannon's 2D matrix multiply

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A: Displace blocks in row i by i
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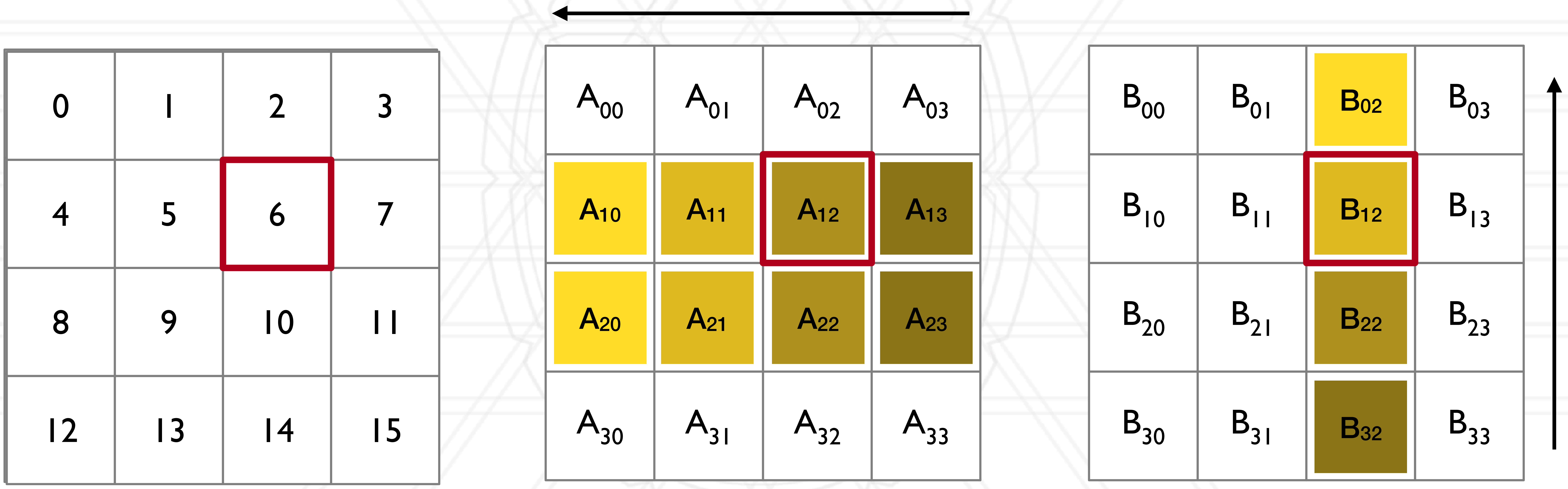

2D process grid

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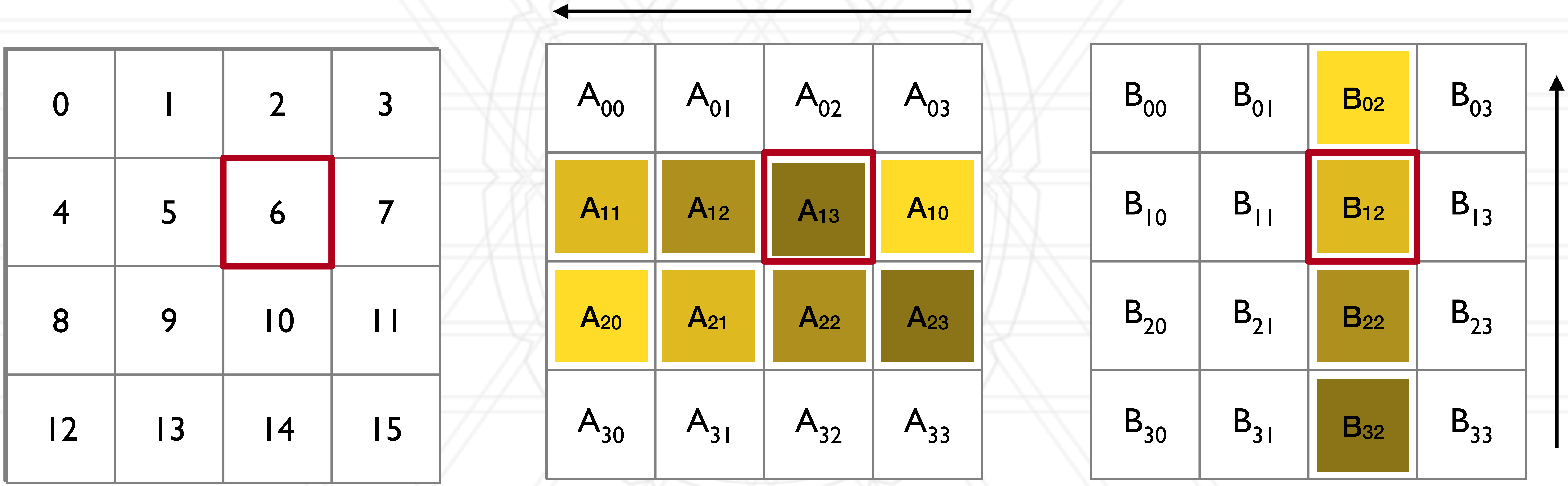

2D process grid

Initial skew in rows

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2D process grid

A_{00}	A_{01}	A_{02}	A_{03}
A_{11}	A_{12}	A_{13}	A_{10}
A_{22}	A_{23}	A_{20}	A_{21}
A_{30}	A_{31}	A_{32}	A_{33}

Initial skew in rows

B_{00}	B_{01}	B_{02}	B_{03}
B_{10}	B_{11}	B_{12}	B_{13}
B_{20}	B_{21}	B_{22}	B_{23}
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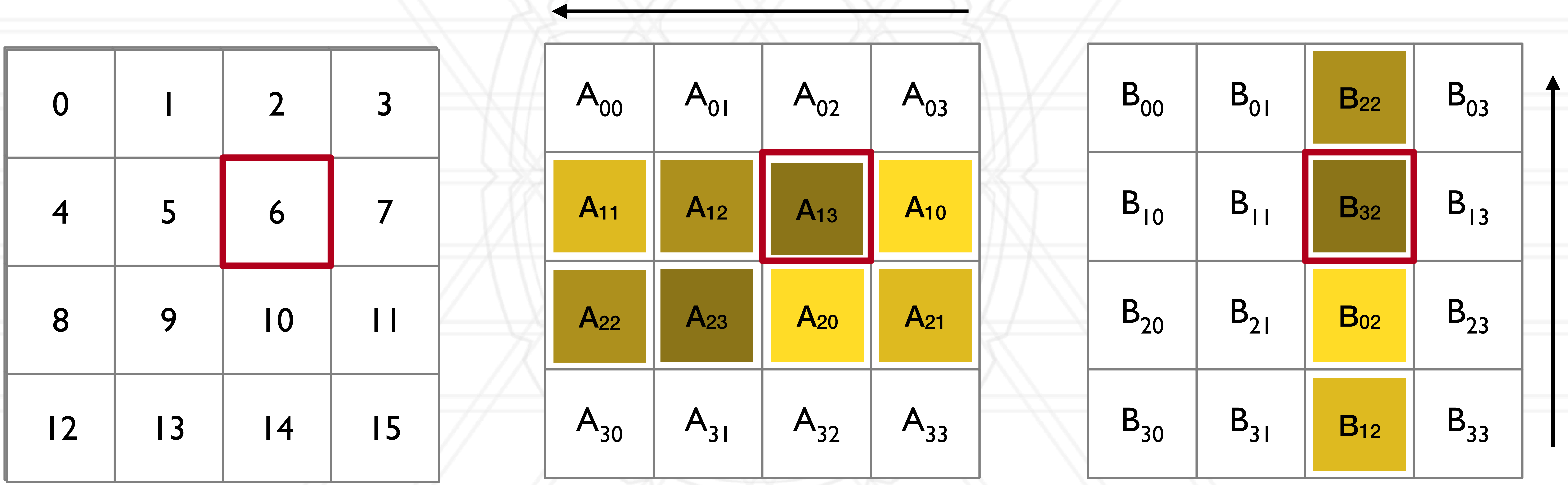

2D process grid | Initial skew in rows | Initial skew in columns

Cannon's 2D matrix multiply

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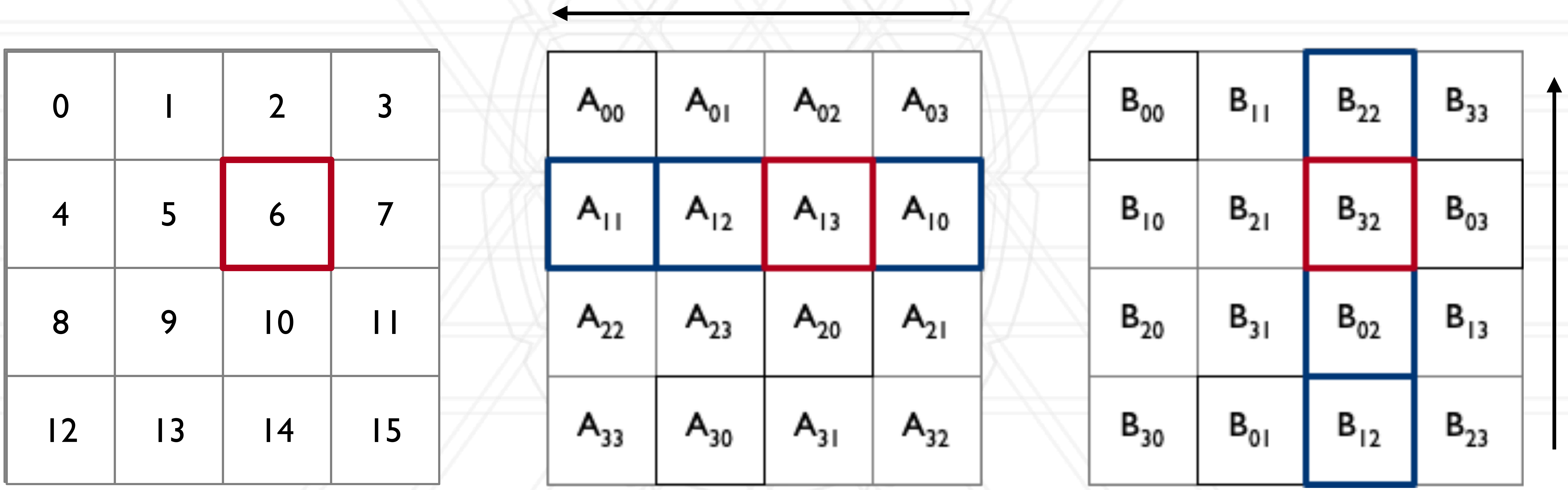

2D process grid

Shift-by-1 in rows

Shift-by-1 in columns

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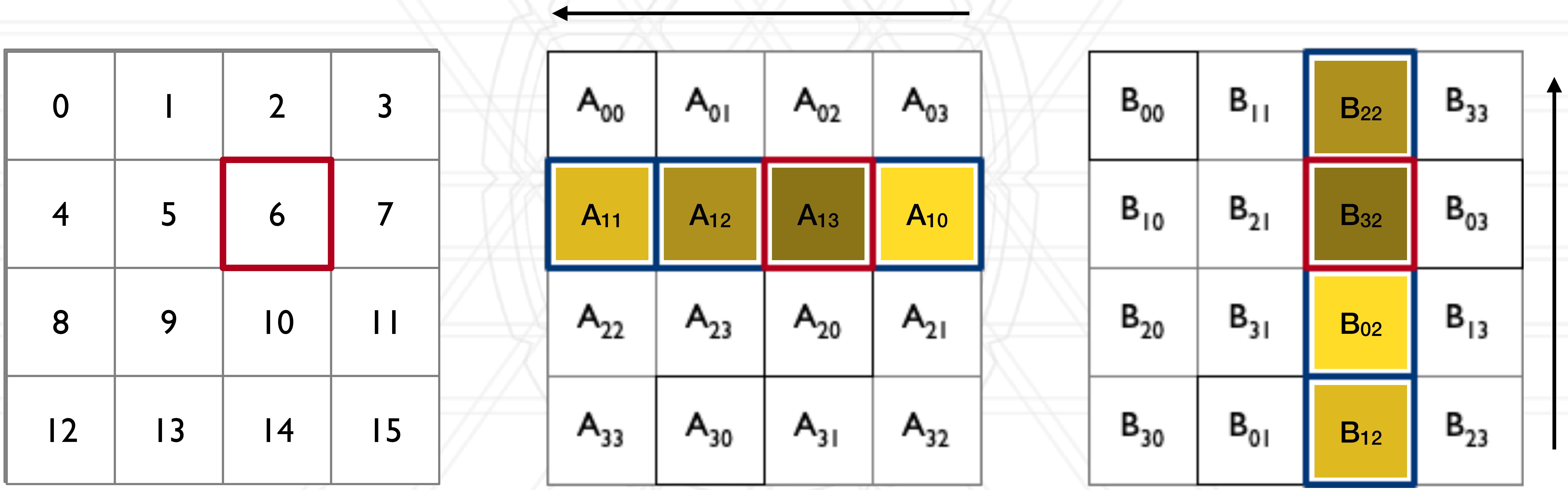

2D process grid

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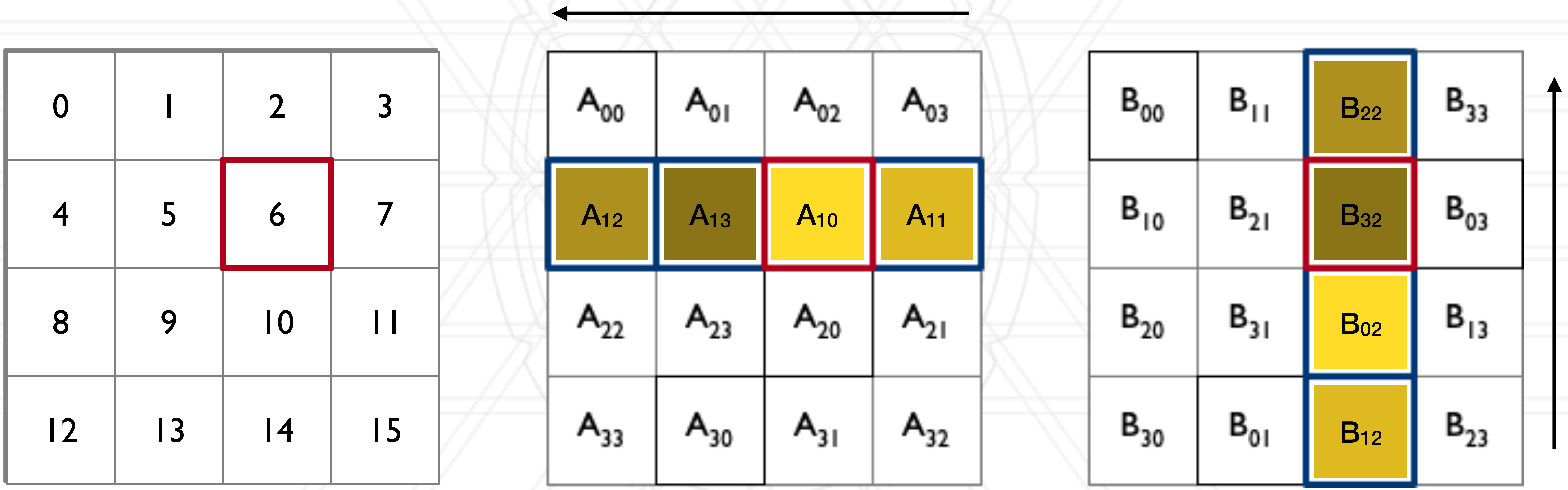

2D process grid

Shift-by-1 in rows

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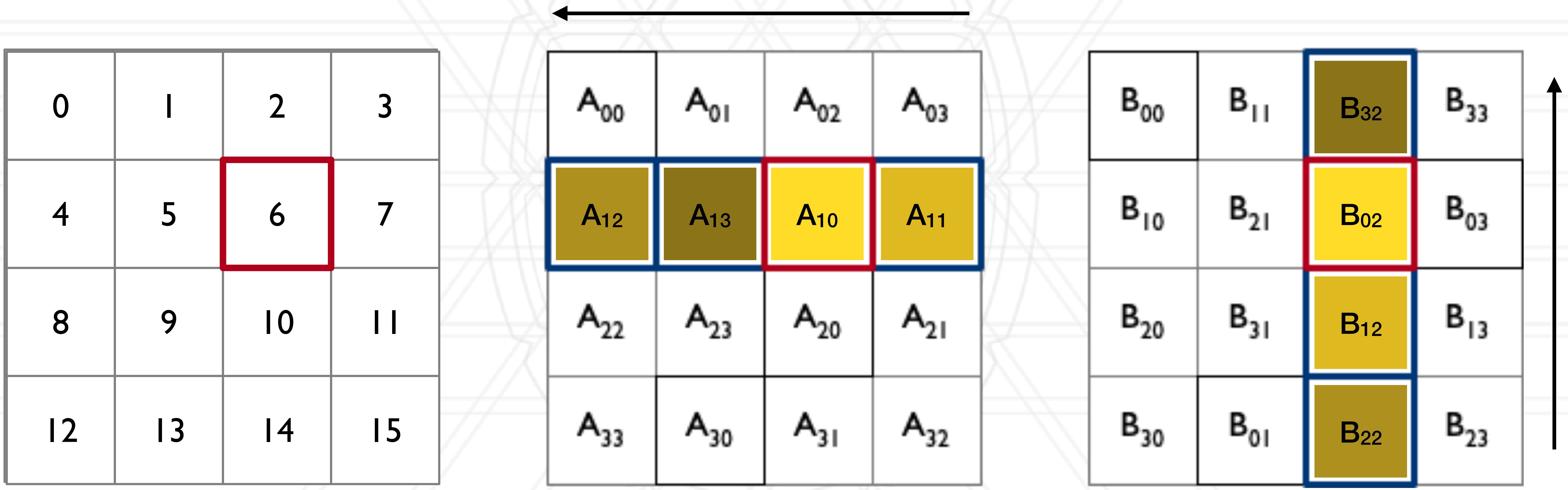

2D process grid

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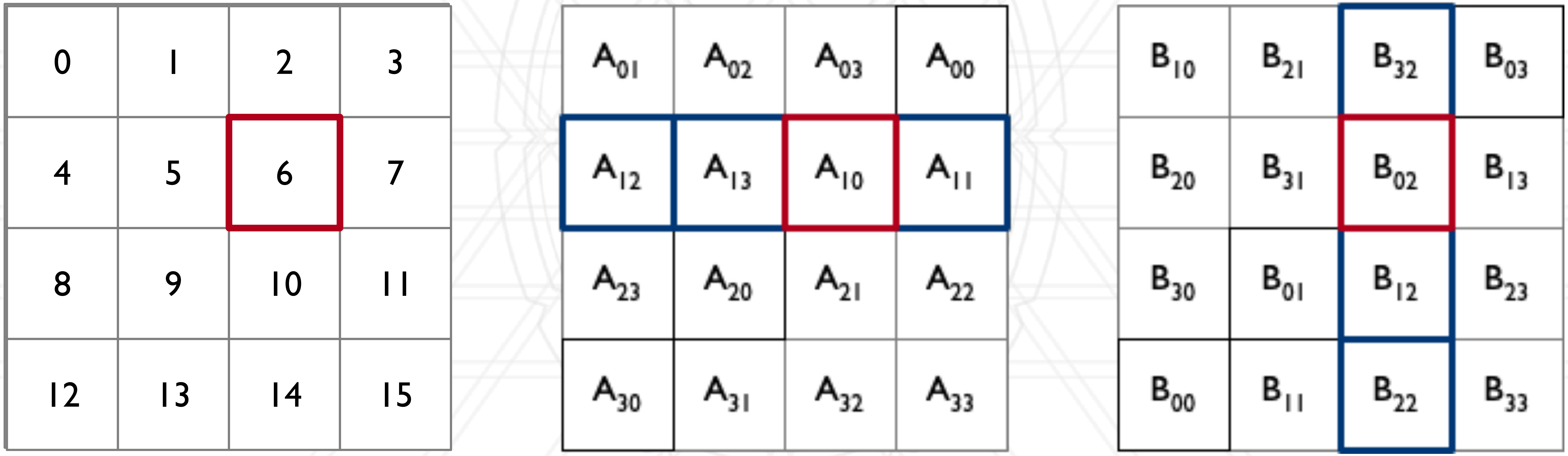

2D process grid

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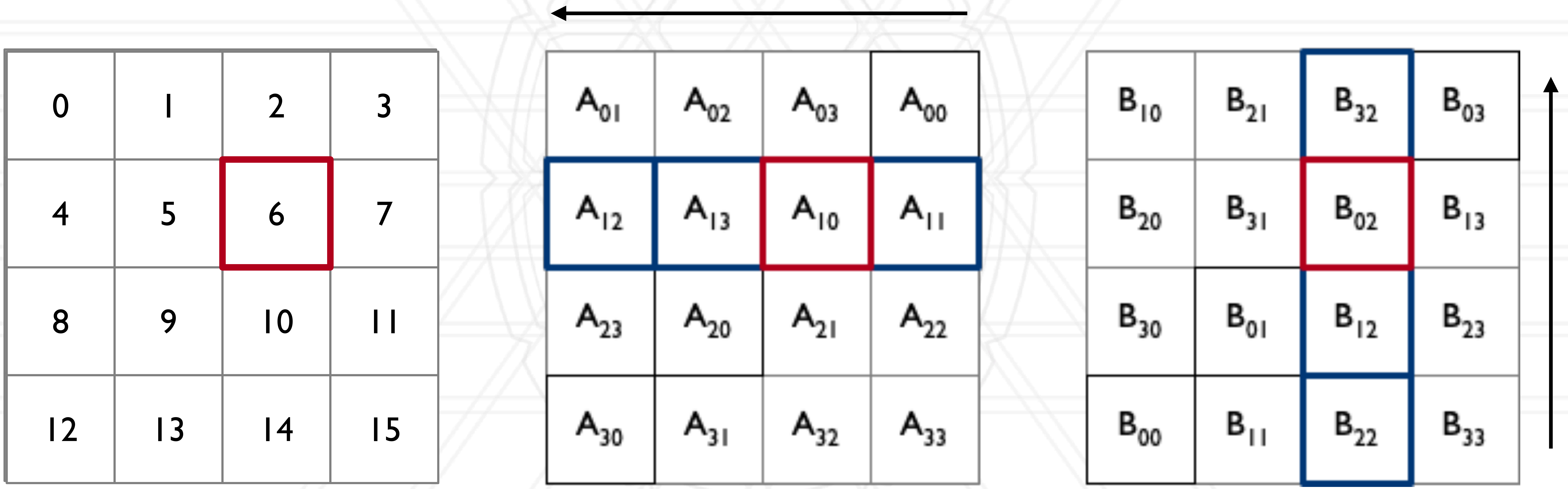


2D process grid | Shift-by-1 in rows | Shift-by-1 in columns

Announcements

- Assignment 2 is due on October 10
- Assignment 3 will be posted on October 10
  - Due on October 18 11:59 pm ET

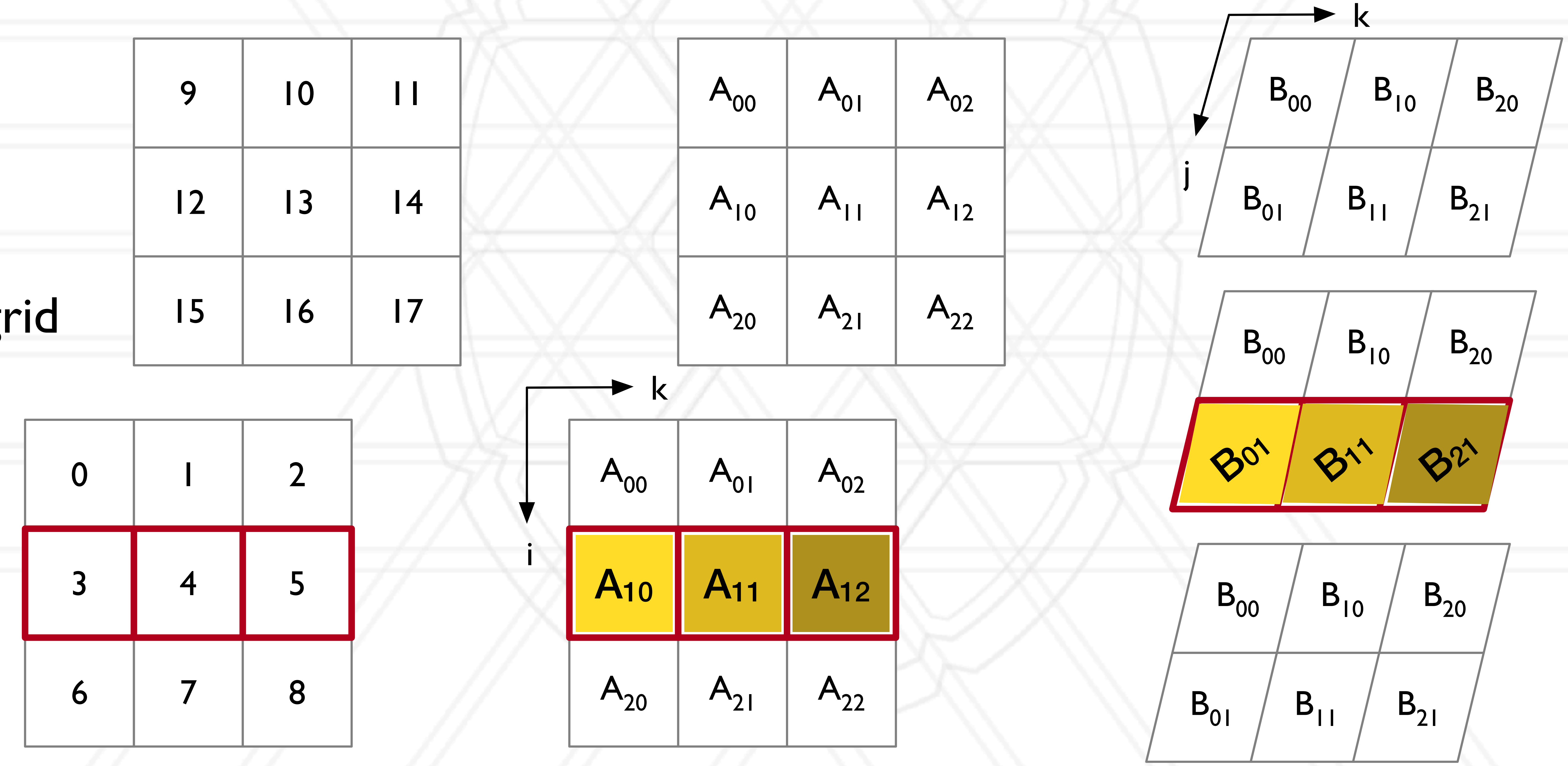
Agarwal's 3D matrix multiply

- Arrange processes in a 3D virtual grid
- Assign sub-blocks of A and B to each process
  - In this algorithm, there are multiple copies of A and B (one in each plane)
- Each process computes a *partial* sub-block of C
- Data movement is done only once before computation and once after computation

Agarwal's 3D matrix multiply

- Copy A to all i-k planes and B to all j-k planes

3D process grid



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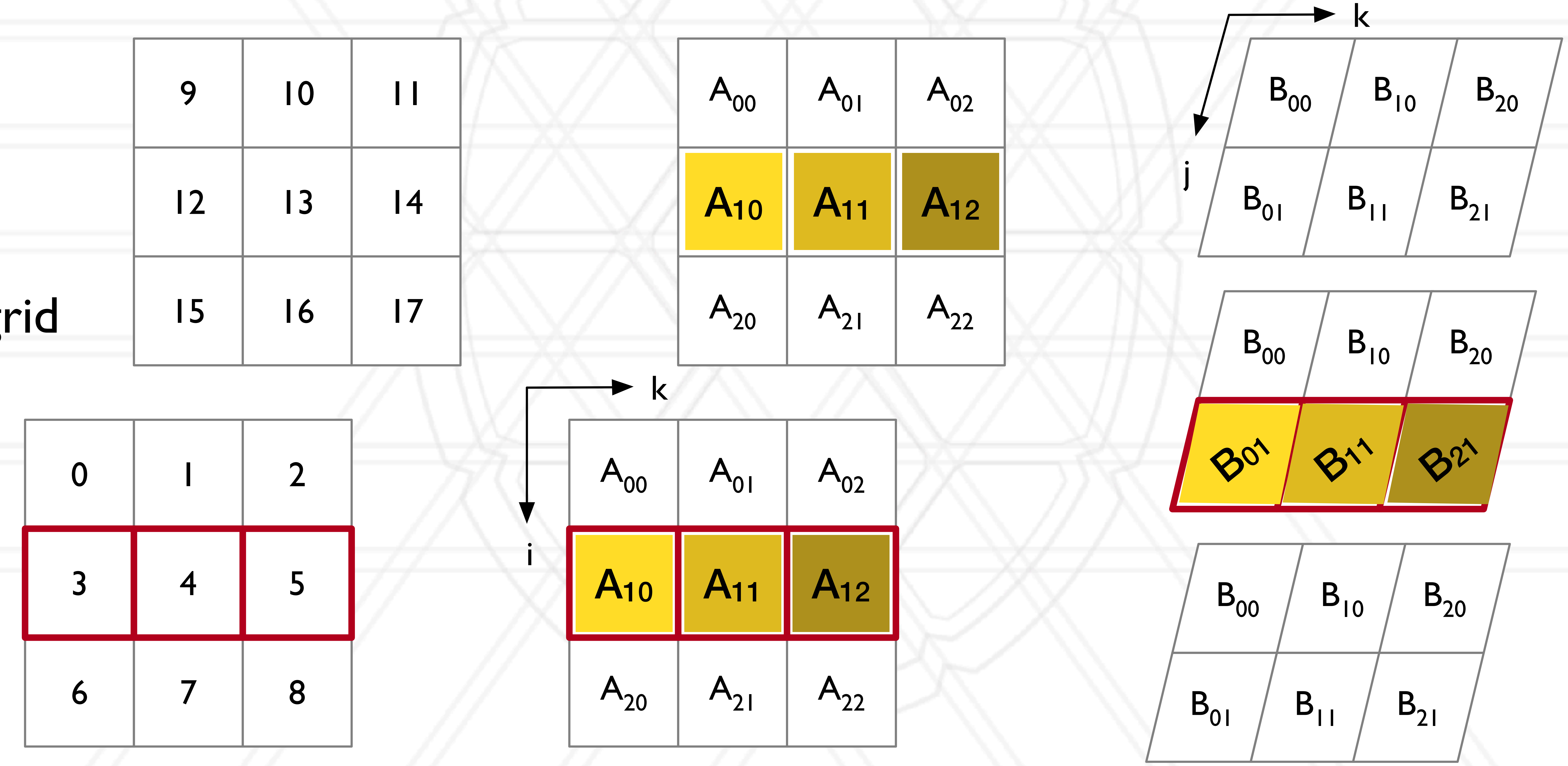
3D process grid



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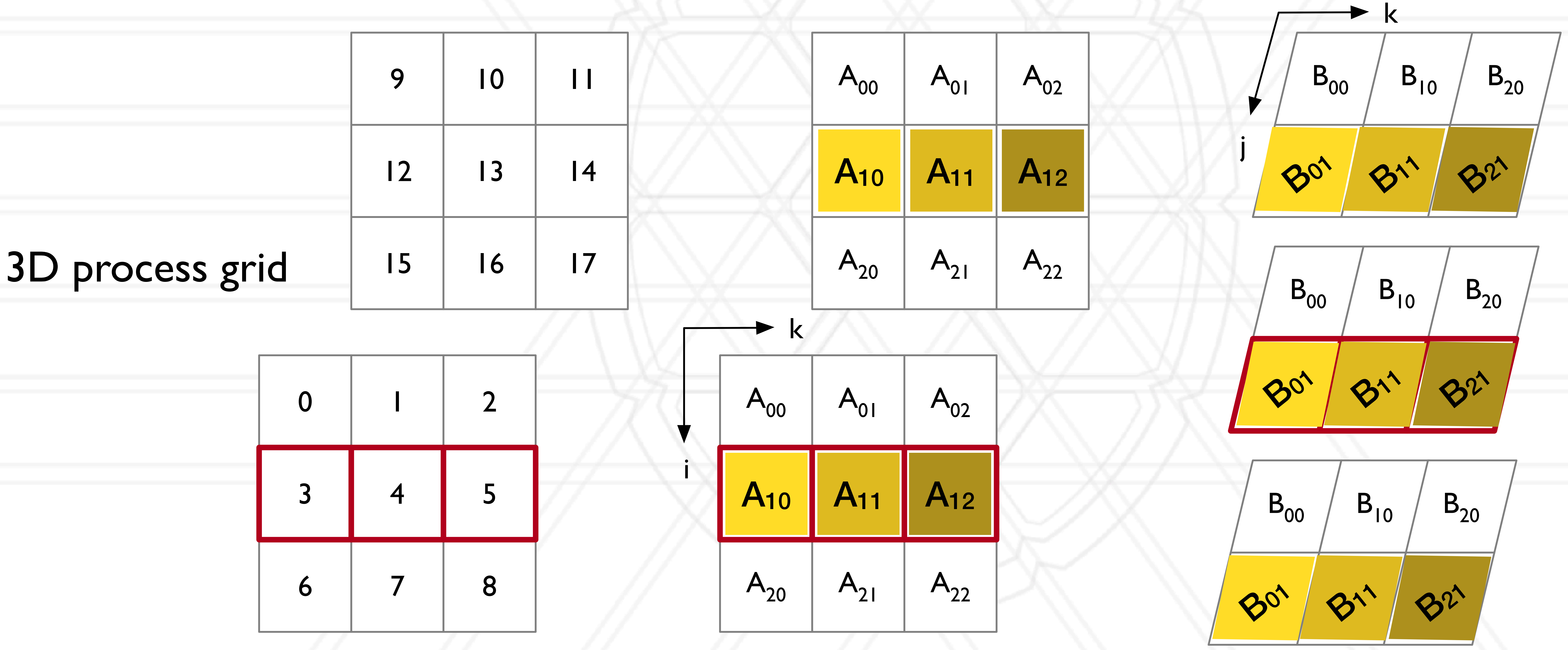
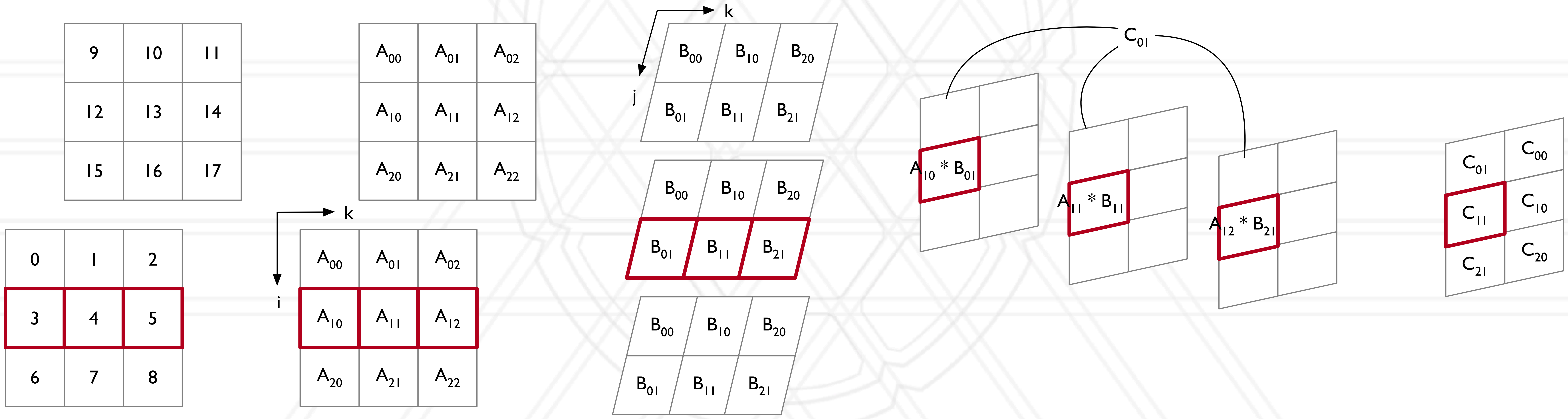


Agarwal's 3D matrix multiply

- Perform a single matrix multiply to calculate partial C
- Allreduce along i-j planes to calculate final result



Communication algorithms

- Reduction
- All-to-all

Types of reduction

- Scalar reduction: every process contributes one number
  - Perform some commutative associate operation
- Vector reduction: every process contributes an array of numbers

Parallelizing reduction

MPI Reduction Algorithms: https://hcl.ucd.ie/system/files/TJS-Hasanov-2016.pdf

Parallelizing reduction

- Naive algorithm: every process sends to the root

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Parallelizing reduction

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- Spanning tree: organize processes in a k-ary tree

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Parallelizing reduction

- Naive algorithm: every process sends to the root
- Spanning tree: organize processes in a k-ary tree
- Start at leaves and send to parents
- Intermediate nodes wait to receive data from all their children

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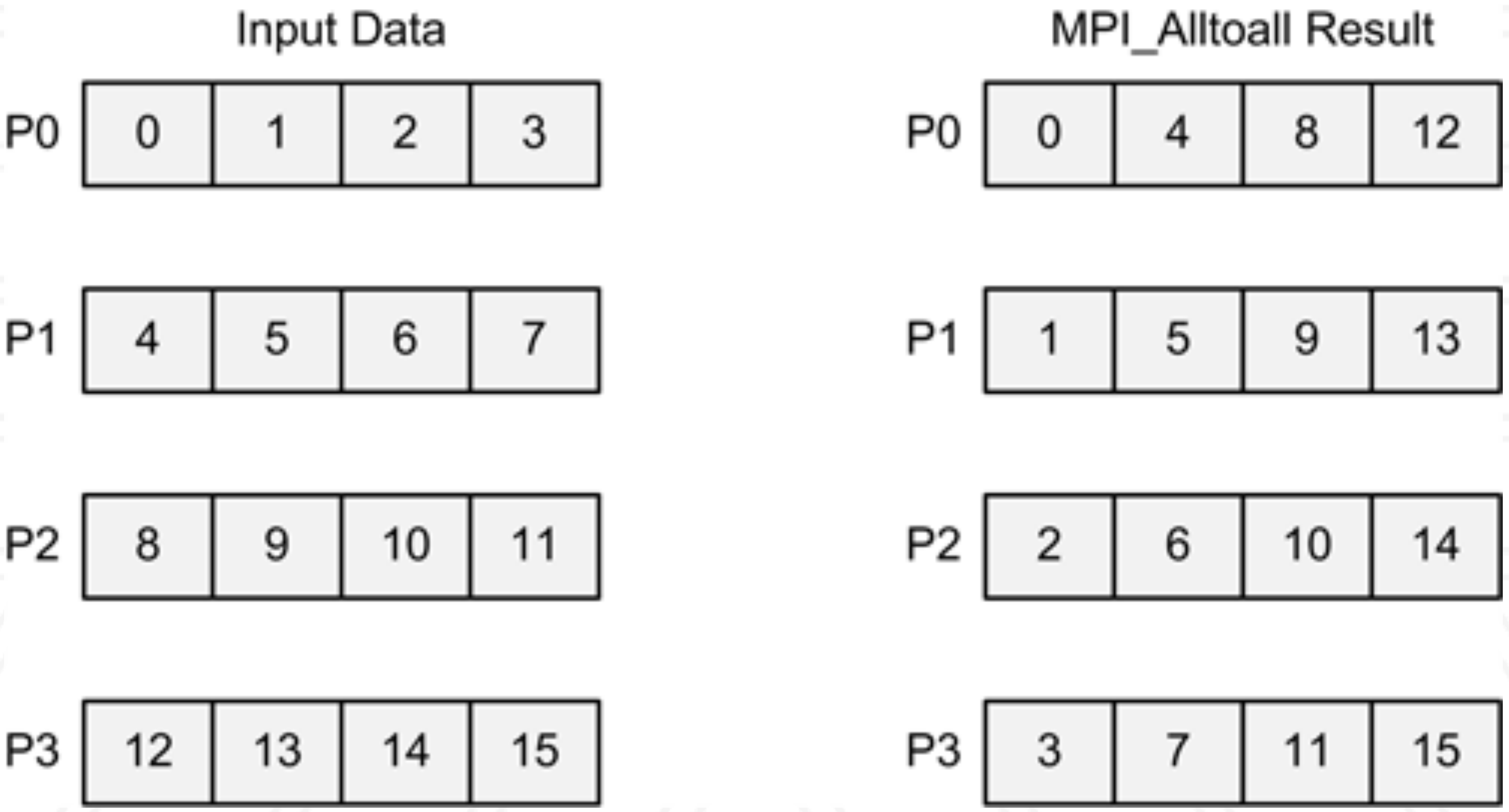
Parallelizing reduction

- Naive algorithm: every process sends to the root
- Spanning tree: organize processes in a k-ary tree
- Start at leaves and send to parents
- Intermediate nodes wait to receive data from all their children
- Number of phases: $\log_k p$

MPI Reduction Algorithms: https://hcl.ucd.ie/system/files/TJS-Hasanov-2016.pdf

All-to-all collective call

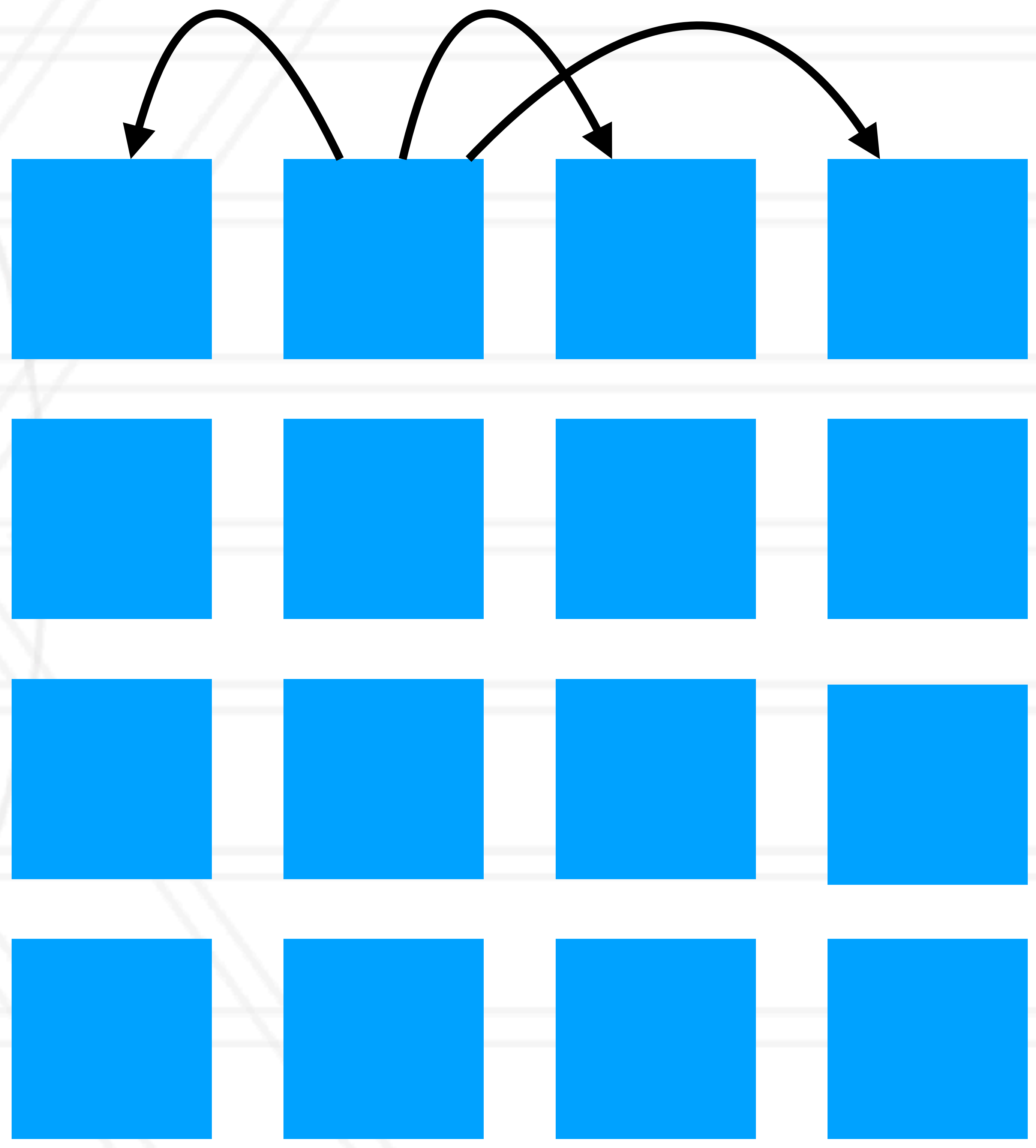
- Each process sends a distinct message to every other process
- Naive algorithm: every process sends the data pair-wise to all other processes



https://www.codeproject.com/Articles/896437/A-Gentle-Introduction-to-the-Message-Passing-Inter

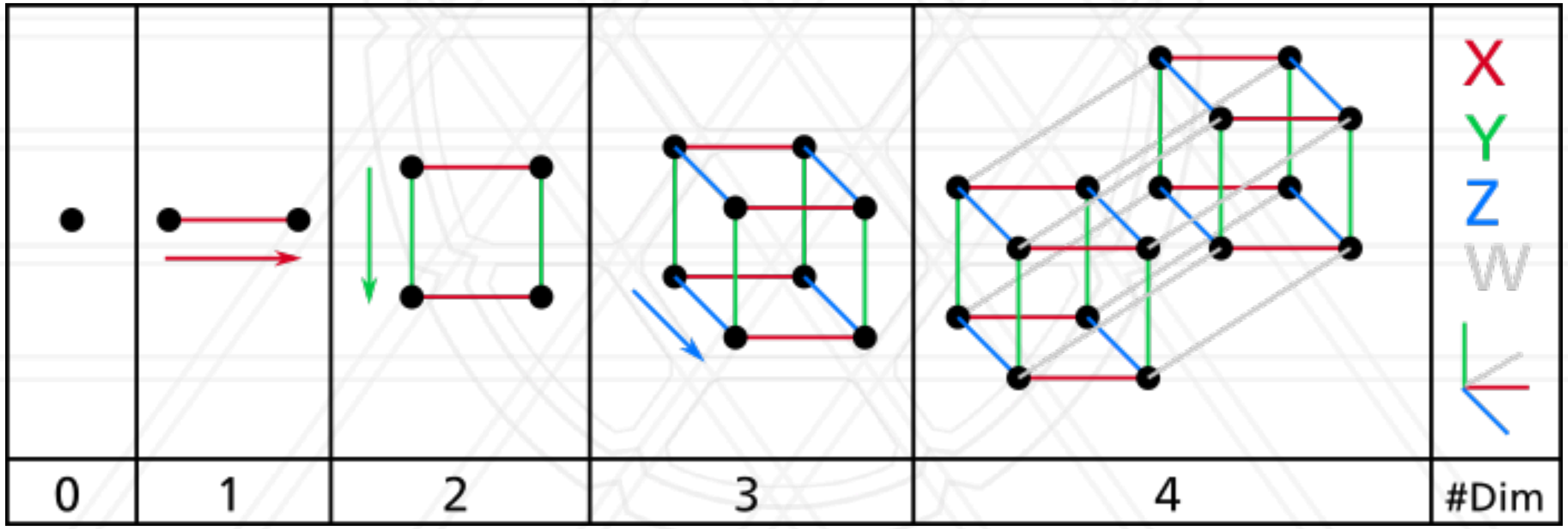
Virtual topology: 2D mesh

- Alternative algorithm: send messages along the rows and columns of a 2D mesh
- Phase 1: every process sends to its row neighbors
- Barrier: wait for phase 1 to complete
- Phase 2: every process sends to column neighbors



Virtual topology: hypercube

- Hypercube is an n-dimensional analog of a square (n=2) and cube (n=3)
- Special case of k-ary d-dimensional mesh



https://en.wikipedia.org/wiki/Hypercube

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