Practice Problems 1

Problem 1. Around 100 A.D., Nicomachus of Gerasa (in ancient Greece) proved the following remarkable identity. For all $n \geq 0$,

$$\sum_{i=1}^{n} i^3 \;=\; \left(\sum_{i=1}^{n} i \right)^2$$

That is $(1^3 + 2^3 + \cdots + n^3) = (1 + 2 + \cdots + n)^2$.

(a) Prove this by induction on n. (Hint: It may be useful to recall the identity for the *arithmetic series*, $\sum_{i=1}^{n} = n(n+1)/2$.)

(b) Explain why Figure 1 provides an pictorial "proof" of this identity. (Hint: The sum of areas of the squares in the figure can be calculated in two ways. One is layer by layer. The other is based on the sizes of squares on the diagonal.)

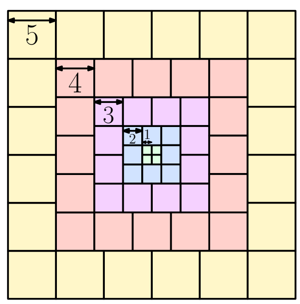

Figure 1: Nicomachus's Theorem

Problem 2. For each of the lists below, put the functions in increasing asymptotic order.

(a)	$(3/2)^n$	$3^{(n/2)}$	$2^{(n/3)}$
(b)	$\lg n$	$\ln n$	$\lg(n^2)$
(c)	$n^{\lg 4}$	$2^{\lg n}$	$2^{(2 \lg n)}$
(d)	$\max(50n^2, n^3)$	$50n^2 + n^3$	$\min(50n^2, n^3)$
(e)	$\lceil n^2/20 \rceil$	$\lfloor n^2/20 \rfloor$	$n^2/20$

In some cases functions may be asymptotically equivalent (that is $f(n)$ is $\Theta(g(n))$). In such cases indicate this by writing $f(n) \approx g(n)$. When one is asymptotically strictly less than the other (that is, $f(n)$ is $O(g(n))$ but $f(n)$ is not $\Theta(g(n))$), express this as $f(n) \prec g(n)$. For example, given the functions:

$$n^2 \qquad n \log n \qquad 30n + 2n \log n,$$

the first function is $\Theta(n^2)$ and the other two are $\Theta(n \log n)$, and therefore the answer would be

$$n \log n \ \approx\ 30n + 2n \log n \ \prec\ n^2.$$

Hint: It may be helpful to review basic logarithm identities and the CMSC 451 Quick Reference Guide.

Problem 3. Show the result of running DFS on the digraph in Fig. 2 using the algorithm given in class. (Start at vertex a. Whenever you have a choice which vertex to visit next, take the lowest vertex in alphabetical order.)

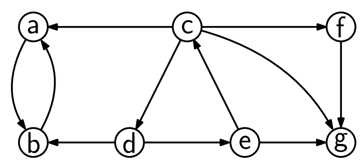

Figure 2: Problem 3: Depth First Search.

Label each node u with its discovery and finish times $(d[u]/f[u])$. As in the lecture notes, show tree edges with solid lines and the other edges with dashed lines. Label these other edges as *forward*, *cross*, or *back* edges. (Show only the **final tree**, not the intermediate results.)

Problem 4. Consider a bipartite graph $G = (V, E)$, where $V = V_1 \cup V_2$ is the partition of the vertex set (meaning that all edges connect a vertex in V_1 to a vertex in V_2). Let $n_1 = |V_1|$ and $n_2 = |V_2|$. As an (exact) function of n_1 and n_2, what is the maximum number of edges that G can have? Briefly justify your answer.

Problem 5. Given an undirected graph $G = (V, E)$, let $\overline{G} = (V, \overline{E})$ denote its *complement*. This means that for any pair of distinct vertices $u, v \in V$, $\{u, v\} \in E$ if and only if $\{u, v\} \notin \overline{E}$. Prove the following lemma.

Lemma: For any undirected graph $G = (V, E)$, at least one of the two graphs G and $\overline{G}$ is connected.