Practice Problems 12

Problem 1. In this problem, we will explore three problems that are closely related to the Directed Hamiltonian Cycle problem (DHC) discussed in class. In each case, present a reduction from a known NP-complete problem to the given decision problem. (You do not need show that the problem is in NP.)

Briefly (in a sentence or so) explain why your reduction runs in polynomial time, and give a careful proof of the correctness of your reduction.

(a) *Directed Hamiltonian Path* (DHP): Given a directed graph $G = (V, E)$, does there exist a path that visits all the vertices exactly once? (See Fig. 1(a).) (**Hint**: Reduction from DHC. Create a start and end vertex for the path.) While it is possible to do this by modifying the 3SAT $\leq_P$ DHC reduction from class, for full credit, reduce directly from DHC.

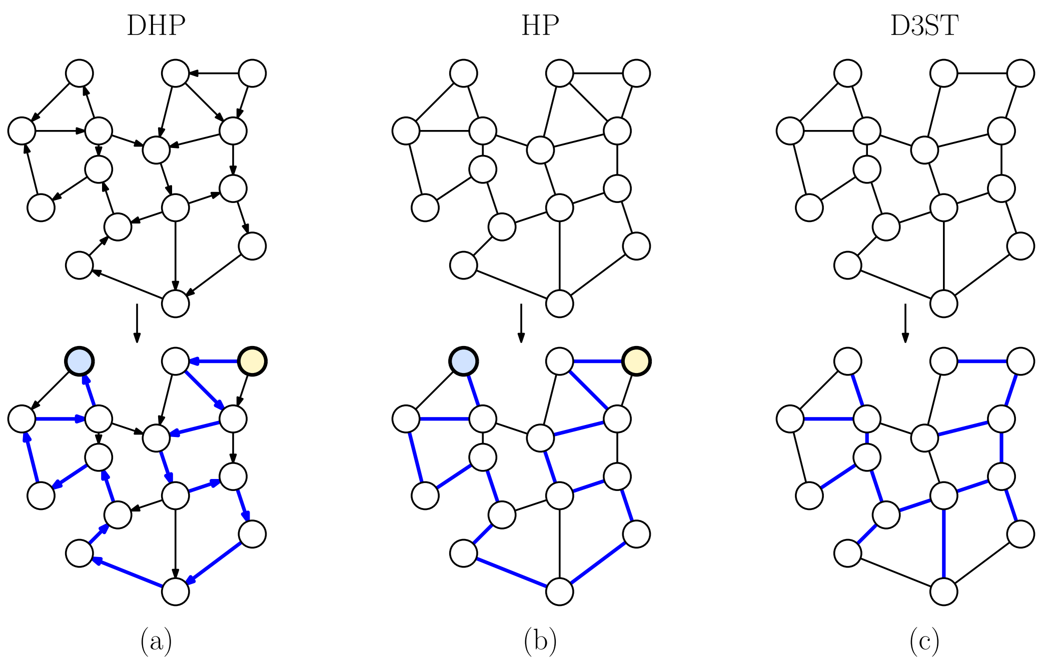


Figure 1: NP-Complete problems related to Hamiltonian cycle.

(b) *Hamiltonian Path* (HP): Given an *undirected* graph $G = (V, E)$, does there exists a path that visits all the vertices exactly once? (See Fig. 1(b).) (**Hint**: Reduction from DHP. One approach involves replacing each vertex of the graph with a small cluster of vertices.)

(c) *Degree-3 Spanning Tree* (D3ST): Given an *undirected* graph $G = (V, E)$, does there exists a spanning tree such that the degree of each vertex in the tree (that is, the number of edges of E' incident on any vertex) is at most three (see Fig. 1(c)). Recall that a *spanning tree* is, a subset $E' \subseteq E$ that induces a connected, acyclic subgraph containing all the vertices of G. (**Hint**: Reduction from HP. This can be done by a local modification to each vertex in the graph.)

Problem 2. Show that the following problem is NP-complete.

Balanced 3-coloring (B3C): Given a graph $G = (V, E)$, where $|V|$ is a multiple of 3, can G can be 3-colored such that the sizes of the 3 color groups are all equal to $|V|/3$. That is, can we assign an integer from $\{1, 2, 3\}$ to each vertex of G such that no two adjacent vertices have the same color, and such that all the colors are used equally often.

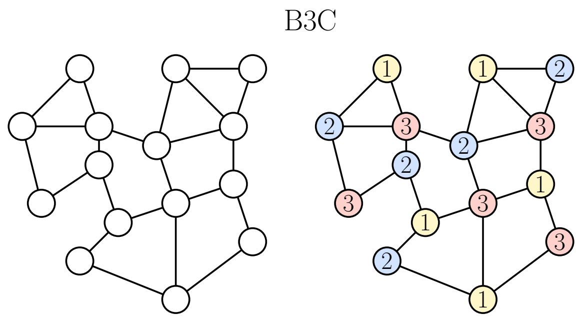


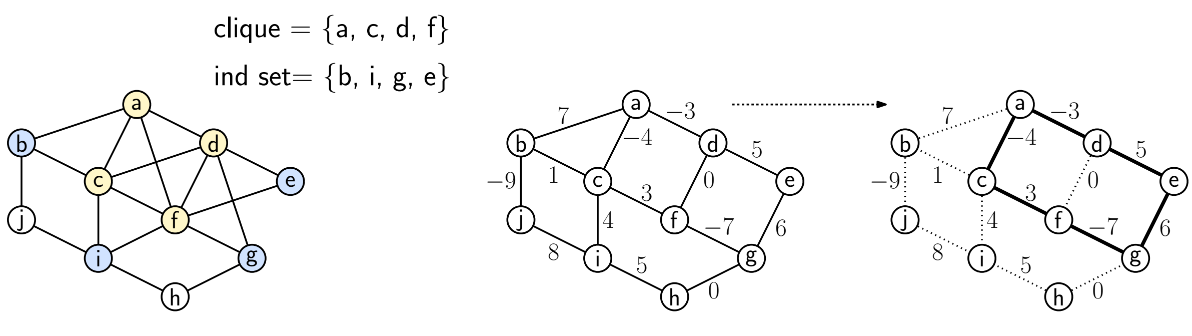
Figure 2: Balanced 3-coloring.

Hint: Reduction from the standard 3-coloring problem (3COL).

Problem 3. A graph is said to be *k-weird* if it has both a clique of size k and an independent set of size k. Given a graph $G = (V, E)$ and an integer k, the k-weird problem (kWP) is that of determining whether G is k-weird. (For example, the graph in Fig. 3(a) is k-weird.)

(a) Show that kWP is in NP.

(b) Prove that kWP is NP-hard. (**Hint**: Reduction from either the clique problem or the independent set problem.)



(a) (b) (c)

Figure 3: The kWP and ZC problems.

Problem 4. Consider the following problem, called the *zero cycle problem* (ZC). You are given an undirected graph $G = (V, E)$ with integer weights on its edges (which may be positive, negative or zero). The question is whether there exists a simple cycle consisting of at least three edges whose total weight is zero? (For example, the graph shown in Fig. 3(b) has the zero cycle shown in Fig. 3(c).)

(a) Show that ZC is in NP.

(b) Prove that ZC is NP-hard. (**Hint**: Reduction from Hamiltonian cycle.)