CMSC 451 Quiz 4

This quiz is closed-book and closed-notes. You may use any algorithms or results given in class. The total point value is 50 points. Good luck!

Problem 1. (15 points) Explanations are not required, but may be provided for the sake of partial credit.

(a) (5 points - 1 point each) You are given a graph with n vertices, m edges, and integer edge weights in the range from 0 to W (represented in binary). For each of the following running times, indicate whether, as a function of input size, it is (SP) strong polynomial time, (WP) weak polynomial time, or (N) neither of these

(i) $O(n^2 + m)$

(ii) $O((n + m) \log n)$

(iii) $O\big(n^{\log m}\big)$

(iv) $O(n + m \cdot W)$

(v) $O(n + m \cdot \log W)$

(b) (10 points - 2 points each) For each of the following claims, state whether it is (T) True, (F) False, or (U) Unknown to science. Let A and B denote two languages.

(i) $\mathrm{P} \subseteq \mathrm{NP}$

(ii) $\mathrm{NP} \subseteq \mathrm{P}$

(iii) If $A \in \mathrm{P}$ and $B \in$ NP-complete then $A \leq_P B$

(iv) If $A \leq_P B$ and $A \notin \mathrm{P}$ then $B \notin \mathrm{P}$

(v) Determining whether a graph is connected is NP-complete

Problem 2. (7 points) You are given a directed network $G = (V, E)$ with start nodes $S = \{s_1, \ldots, s_k\}$ and terminal nodes $T = \{t_1, \ldots, t_\ell\}$. Present an efficient algorithm to determine the *smallest subset of edges* which, if removed, *blocks all paths* from any start node to any terminal node. (In Fig 1, the answer is the 4 dashed edges on the right.)

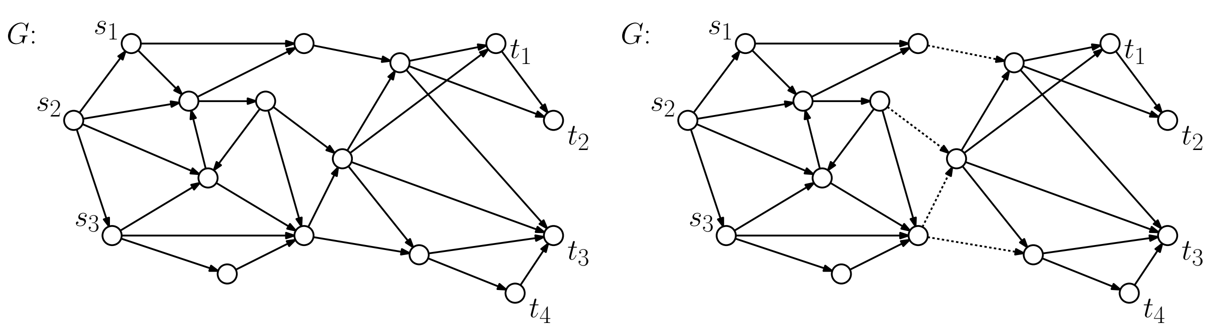

Figure 1: Eliminating edges to separate starts from terminals.

Briefly explain your solution. (A formal proof of correctness is not required. **Hint:** Reduction to network flow. You may use any network flow algorithm from class.)

Problem 3. (8 points) In the *3-colorable decision problem* (3Col), you are given an undirected graph $G = (V, E)$, and the question is whether the vertices of G be labeled with three colors (say, 1, 2, 3) such that no edge is incident to two vertices of the same color.

Suppose that you have a function $\mathsf{3Col}(G)$, which answers the 3-colorable decision problem in polynomial time and returns true if G is 3-colorable and false otherwise (see Fig. 2).

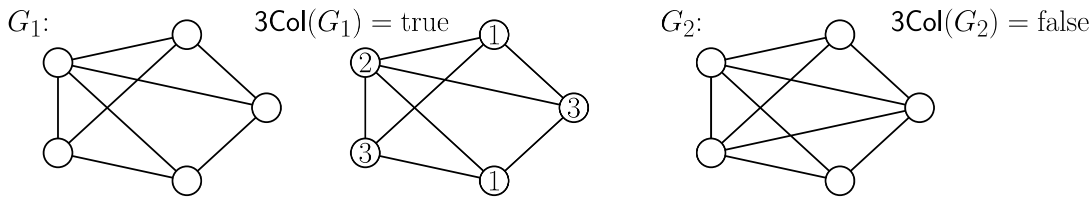

Figure 2: 3-Colorability.

Show that if G is 3-colorable, then it is possible to use this function (as a black box) to determine a valid assignment of colors (say, 1, 2, 3) to all the vertices in polynomial time.

Problem 4. (10 points) An *independent set* in a graph $G = (V, E)$ is a subset $V' \subseteq V$ of vertices such that for all $u, v \in V'$, $\{u, v\} \notin E$. In the *Ultra-Degree Independent Set problem* (UDIS), you are given an undirected graph $G = (V, E)$ and a positive integer k, and you want to know whether there exists an independent set V' of size k such that each vertex of V' is of degree at least $2k$ in G. (This is the same as one of the practice problems, but the degree is $2k$, rather than k.)

For example, the graph in Fig. 3 has a UDIS for $k = 3$, shown as the shaded vertices since no two are adjacent and each has degree at least $2k = 6$.

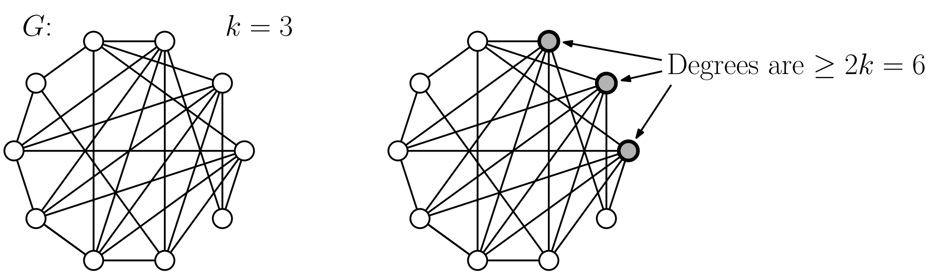

Figure 3: This graph has a UDIS for $k = 3$, but not for $k = 4$.

Prove that IS $\leq_P$ UDIS, where IS is the standard independent set problem. Give a clear explanation of your reduction and explain it briefly. (A full correctness proof is not required. You do not *not* need to show that UDIS is in NP.)

Problem 5. (10 points) You are given a sequence of n buckets labeled 1 through n holding a total of m balls. You are given two sequences of nonnegative integers $A = \langle a_1, \ldots, a_n \rangle$ and $B = \langle b_1, \ldots, b_n \rangle$, where $\sum_i a_i = \sum_i b_i = m$. Initially, the ith bucket holds a_i balls, for $1 \leq i \leq n$. Your objective is to determine whether it is possible to redistribute the balls so that after, the ith bucket holds b_i balls, for $1 \leq i \leq n$. The redistribution must satisfy the following requirements:

- Each ball can remain in the same bucket, or it can move at most two buckets away, either left or right. (That is, a ball in bucket i can stay in bucket i or move to any of the buckets $\{i-2, i-1, i+1, i+2\}$, provided that the index is in the range $[1, n]$.)
- At most 2 balls can move from any one bucket to any *different* bucket.
- For each bucket, at least 1 of its original balls must remain in this bucket. (You may assume $a_i \geq 1$, for all i.)

An example of an input and a redistribution is shown in Fig. 4.

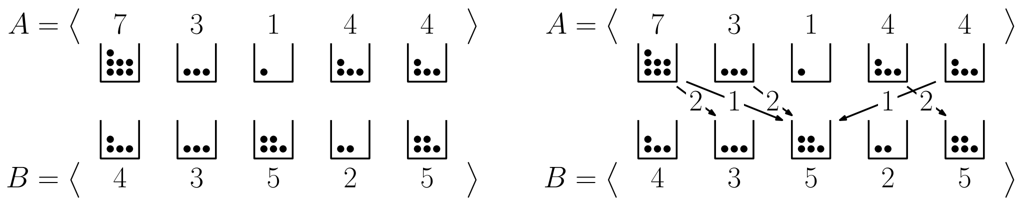

Figure 4: Ball redistribution.

(a) Explain how to model this problem as a circulation problem (with lower and upper edge capacities). Briefly explain your construction. (We do *not* want a full correctness proof.)

(b) Illustrate your construction on the following input: $A = \langle 5, 3, 1 \rangle$ and $B = \langle 2, 2, 5 \rangle$. (Just present the construction of the circulation network, you do not need to show the final flow values.)