CMSC 451 Quiz 5

This quiz is closed-book and closed-notes. You may use any algorithms or results given in class. The total point value is 50 points. Good luck!

Problem 1. (9 points) Explanations are not required, but may be provided for partial credit.

  (a) (5 points) A graph G with n vertices has a *vertex cover* of size k. Which of the following inferences can be made? (Select all that apply)

   (i) G has an *independent set* of size $n - k$
   (ii) G cannot have an *independent set* of size $> n - k$
   (iii) G cannot have a *clique* of size $> k$
   (iv) The complement graph $\overline{G}$ has a *clique* of size k
   (v) The complement graph $\overline{G}$ has a *clique* of size $n - k$

  (b) (2 points) Aliens show us that there is an $O(n^k)$-time algorithm that determines whether a graph with n vertices has a clique of size k. True or False: This implies that P = NP.

  (c) (2 points) Suppose that $A \leq_P B$, and there is a factor-2 approximation to problem B. Which of the following can you infer from this:

   (i) There is a factor-2 approximation for A
   (ii) There is a constant-factor approximation for A, but the factor might not be 2
   (iii) We cannot infer anything about our ability to approximate A

Problem 2. (16 points) This problem involves variations of Hamiltonian Cycle and Hamiltonian Path for undirected and directed graphs. Recall that in each case, we want to know whether there exists a simple cycle or simple path, respectively, that visits all the vertices. Let UHC and UHP denote the undirected versions of these problems, and let DHC and DHP denote the directed versions (see Fig. 1).
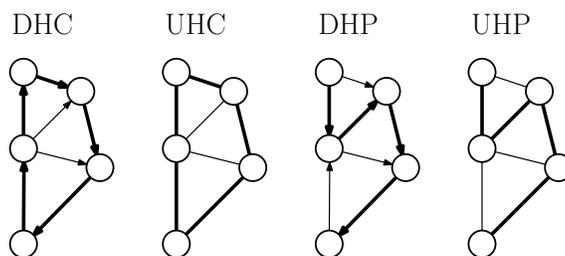

Figure 1: Subset sum approximation.

  (a) (8 points) Prove that DHC $\leq_P$ DHP. Present a reduction function, and briefly explain it correctness. (A couple of sentences suffice. We don't want a full "iff" proof.) Illustrate your reduction on a small example.

(b) (8 points) Prove that UHC $\leq_P$ UHP. Again, present a reduction function, and briefly explain it correctness. (Again, keep it brief.) Illustrate your reduction on a small example.

Problem 3. (7 points) The *set cover* optimization problem is: Given a pair (X, S), where $X = \{x_1, \ldots, x_n\}$ is a finite set and a $S = \{s_1, \ldots, s_m\}$ is a collection of subsets of X, find a *minimum sized* collection of these sets C whose union equals X.

Consider a special variant of this problem, where each element of X occurs in *at most three sets of S*.

Present a polynomial-time approximation algorithm for this variant that achieves an approximation ratio bound of 3. (That is, for any input, your algorithm returns a cover whose size is larger than the optimum by a factor of at most 3. You may assume a valid cover exists.)

Briefly explain why your algorithm achieves the factor-3 approximation ratio. (**Hint:** Similar to the vertex-cover approximation algorithm.)

Problem 4. (12 points) Recall the two TSP heuristics presented in class (Twice-around and Christofides). To simplify calculations, assume that lengths and distances are measured by the L_1 (or *Manhattan*) metric, where the distance between two points is defined to be the sum of the horizontal plus vertical differences between them (see Fig. 2 left).

Consider the point set $P = \{a, b, \ldots, j\}$ and associated minimum spanning tree (MST) shown in Fig. 2 right. Every edge has a length of 1.
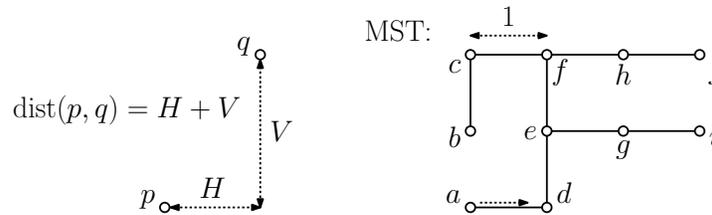

Figure 2: TSP heuristics.

(a) What is the total length of the MST shown in the figure?

(b) List the points in the order they are visited according to the twice-around tour. (For consistency, start at a and end with a. Whenever there is choice of the next point to visit, select the lowest in alphabetical order. Assume that short-cutting is applied, so no point is visited twice.)

(c) What is the total length (in the Manhattan distance) of the tour from (b)?

(d) Christofides algorithm involves computing a minimum-weight matching of the odd-degree vertices of the MST. List the edges in this matching.

(e) (Prob. 4 continued) What is the total weight of this matching?

(f) List the points in the order they are visited according to the Christofides heuristic. (Start at a, but after that you may visit the points in any valid order.)

(g) What is the total length of your tour from (f)?

Problem 5. (6 points) Consider the bucketed version of the subset-sum (SS) approximation algorithm. Let the target value be $t = 15$, and let us assume that the buckets are the intervals $[0]$, $[1]$, $[2,3]$, $[4,5]$, $[6,8]$, $[9,11]$, and $[12,16]$ (see Fig. 3). Let the current list be $L_{i-1} = \{0, 3, 4, 10, 14\}$.
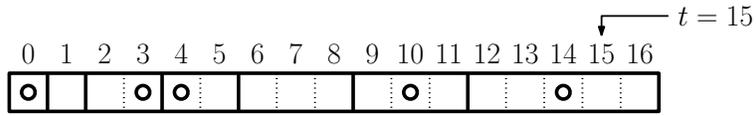

Figure 3: Subset sum approximation.

Let $x_i = 2$. Show the list L_i that results from the next step of the algorithm, both *before* and *after* the pruning process.