Solutions to Practice Problems 1

Solution 1:

(a) For any $n \geq 0$, let $L(n) = \sum_{i=1}^{n} i^3$ and let $R(n) = (\sum_{i=1}^{n} i)^2$. It suffices to show that $L(n) = R(n)$. In our proof, we will make use of a standard fact on the arithmetic series, $A(n) = \sum_{i=1}^{n} i = n(n+1)/2$.

For the basis case, $n = 0$, we have $L(n) = 0$ and $R(n) = 0$, so clearly, $L(n) = R(n)$. For the induction step, let's fix a value $n \geq 1$, and assume as the induction hypothesis that $L(n-1) = R(n-1)$. We will prove that $L(n) = R(n)$.

By definition and straighforward manipulations, we have

$$L(n) \;=\; \sum_{i=1}^{n} i^3 \;=\; \left(\sum_{i=1}^{n-1} i^3\right) + n^3 \;=\; L(n-1) + n^3. \tag{1}$$

and

$$R(n) \;=\; \left(\sum_{i=1}^{n} i\right)^2 \;=\; \left(\left(\sum_{i=1}^{n-1} i\right) + n\right)^2 \;=\; \left(\sum_{i=1}^{n-1} i\right)^2 + 2n\sum_{i=1}^{n-1} i + n^2.$$

We can rewrite $R(n)$ as follows, using the above fact about the arithmetic series

$$\begin{aligned} R(n) \;&=\; R(n-1) + 2nA(n-1) + n^2 \;=\; R(n-1) + 2n\frac{n(n-1)}{2} + n^2 \\ &=\; R(n-1) + n^3. \end{aligned} \tag{2}$$

By the induction hypothesis, $L(n-1) = R(n-1)$, which implies that $L(n-1) + n^3 = R(n-1) + n^3$. Combining this with Eqs. (1) and (2) yields

$$L(n) \;=\; L(n-1) + n^3 \;=\; R(n-1) + n^3 \;=\; R(n),$$

and therefore $L(n) = R(n)$, completing the proof.

(b) Define the *size* of a square to be length of one of its sides. Let us group the squares of the figures into groups of equal size. Working layer-by-layer from the inside to outside, the square sizes grow as $s(i) = i = 1, 2, 3, \ldots, n$. The number of squares within the various size groups are $m(i) = 4i = 4, 8, 12, \ldots, 4n$. Using the fact that a square of side length s has area s^2, we can compute the area of the large square by summing the areas of the squares of each size group:

$$\sum_{i=1}^{n} m(i)s(i)^2 \;=\; \sum_{i=1}^{n} (4i)i^2 \;=\; 4(1^3 + 2^3 + \cdots + n^3) \;=\; 4\sum_{i=1}^{n} i^3. \tag{3}$$

The other way to compute the area is to take the square of large square's side length, which we denote by $S(n)$. We'll compute the side length by summing the side lengths of the squares that lie along the diagonal. Note that these squares do not overlap horizontally or vertically,

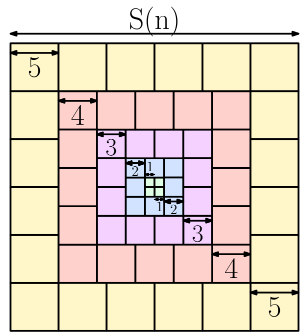

Figure 1: Nicomachus's Theorem

and they cover the entire width of the large square. It follows that the side length of the large square is (see Fig. 1)

$$S(n) \;=\; (n + \ldots + 3 + 2 + 1) + (1 + 2 + 3 + \ldots + n) \;=\; 2(1 + 2 + 3 + \ldots + n) \;=\; 2\sum_{i=1}^{n} i.$$

Thus, the total area of the square is Applying this to the vertical side as well, it follows that the area of the entire square is the square of this quantity, or

$$S(n)^2 \;=\; \left(2\sum_{i=1}^{n} i\right)^2 \;=\; 4\left(\sum_{i=1}^{n} i\right)^2. \tag{4}$$

The areas computed in Eqs. (3) and (4) must be equal, and eliminating the common factor of 4 yields

$$\sum_{i=1}^{n} i^3 \;=\; \left(\sum_{i=1}^{n} i\right)^2,$$

as desired.

Solution 2: Here are the asymptotic relationships between the various formulas. Justifications are given below.

(a)	$2^{(n/3)}$	$\prec$	$(3/2)^n$	$\prec$	$3^{(n/2)}$
(b)	$\lg n$	$\approx$	$\ln n$	$\approx$	$\log(n^2)$
(c)	$2^{\lg n}$	$\prec$	$n^{\lg 4}$	$\approx$	$2^{(2\lg n)}$
(d)	$\min(50n^2, n^3)$	$\prec$	$50n^2 + n^3$	$\approx$	$\max(50n^2, n^3)$
(e)	$\lfloor n^2/20 \rfloor$	$\approx$	$n^2/20$	$\approx$	$\lceil n^2/20 \rceil$

(a) We use the fact that $a^{bc} = (a^b)^c$, and hence $2^{(n/3)} = (2^{1/3})^n \approx 1.26^n$, $3^{(n/2)} = (3^{1/2})^n \approx 1.73^n$, and finally $(3/2)^n = 1.5^n$. If $x < y$, then $y^n/x^n = (y/x)^n$, which tends to infinity, and therefore $x^n \prec y^n$.

(b) The first two are equivalent because changing the base of a logarithm only changes the function by a constant factor. (Recall that $\log_a b = \log_c b/(\log_c a)$.) The last one can be rewritten as $\lg(n^2) = 2\lg n \approx \lg n$.

(c) $2^{\lg n} = n$, $n^{\lg 4} = n^2$, and finally $2^{2\lg n} = (2^{\lg n})^2 = n^2$.

(d) A nice fact to use here is that $\max(f, g) \approx f + g$, which follows from the observation that

$$\max(f,g) \;\leq\; f+g \;\leq\; 2\cdot\max(f,g).$$

This shows the second relationship. To see the first observe that for all sufficiently large n, $\min(50n^2, n^3) = 50n^2 = \Theta(n^2)$ for all sufficiently large n, whereas the middle term is $\Theta(n^2)$.

(e) The floor and ceiling are within an additive constant of 1 of the original value. Since all three values tend to infinity, this difference is negligible in the limit.

Solution 3: See Fig. 2. Each vertex is labeled with its discovery and finish times ($d[u]/f[u]$). Tree edges are solid and the other edges are dashed and labeled by their type.

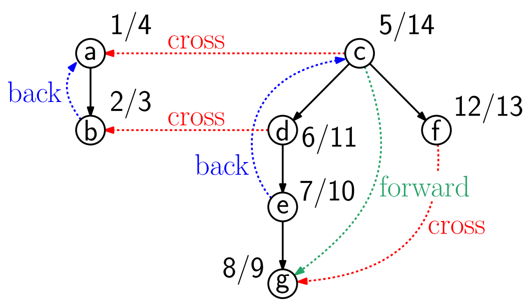

Figure 2: Problem 3: Depth First Search.

Solution 4: A bipartite graph $G = (V_1 \cup V_2, E)$, where $n_1 = |V_1|$ and $n_2 = |V_2|$ can have at most $n_1 n_2$ edges. The reason is that each of the n_1 vertices of V_1 can be adjacent to at most all the n_2 vertices of V_2.

Solution 5:

Lemma: For any graph $G = (V, E)$, at least one of the two graphs G and $\overline{G}$ is connected.

Proof: If the graph G is connected, then the lemma holds. Otherwise, let V_1 denote any connected component of G and let $V_2 = V \setminus V_1$ denote the remaining vertices. Clearly, there is no edge between V_1 and V_2 in G, which implies that in $\overline{G}$ there is an edge between every pair of vertices (u_1, u_2) where $u_1 \in V_1$ and $u_2 \in V_2$.

To show that $\overline{G}$ is connected, consider any two vertices u and v. If u and v are in different subsets V_1 and V_2, then by the above observation they are connected by an edge. On the other hand if both are in the same set (say, V_1) then there is a path $\langle u, w, v\rangle$, where w is any vertex from the other subset (V_2). Therefore, $\overline{G}$ is connected.