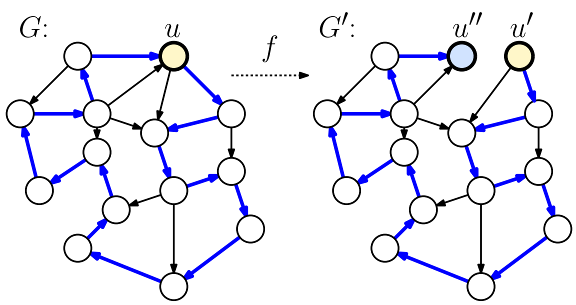
Solutions to Practice Problems 12

Solution 1:

(a) A Hamiltonian cycle passes through every vertex of the graph. The trick is to take any vertex and split it into two vertices, such that all outgoing edges from the original vertex leave from one these vertices and all incoming edges to the original vertex feed into the other. Now, any Hamiltonian cycle in the original graph will be converted into a Hamiltonian path by starting and ending at the split vertices.

Here is the formal reduction. Given a directed graph $G = (V, E)$ for DHC, take any vertex u and replace it with two new vertices, u' and u''. All the edges outgoing from the original u now are outgoing from u' (by replacing each edge (u, v) with (u', v)), and all the edges incoming to the original u now are incoming to u'' (by replacing each edge (v, u) with (v, u'')). Let G' be the resulting graph (see Fig. 1(a)). Correctness is established by the following claim.



(a) (b)

Figure 1: DHC to DHP reduction.

Claim: G has a Hamiltonian cycle if and only if G' has a Hamiltonian path.

Proof: ($\Rightarrow$) Suppose that G has a Hamiltonian Cycle $\langle u_0, u_1, \ldots, u_{n-1}\rangle$. Since the cycle visits all vertices of G, we may assume without loss of generality that $u_0 = u$ (where u is the split vertex in the reduction). It follows that G' has the path $\langle u', u_1, \ldots, u_{n-1}, u''\rangle$. Clearly, this is a Hamiltonian path in G' (see Fig. 1(b)).

($\Leftarrow$) Suppose that G' has a Hamiltonian Path. The path must start at u' and end at u'', because these vertices have only outgoing and incoming edges, respectively. Therefore, the path must have the form $\langle u', u_1, \ldots, u_{n-1}, u''\rangle$, for some sequence $u_1, \ldots, u_{n-1}$ that forms a simple path in G'. We can convert this into a cycle in G by replacing u' and u'' with the single node u. Since the Hamiltonian path visits all the vertices of G', this cycle visits all the vertices of G. Therefore, G has a Hamiltonian Cycle.

(b) Reducing from a directed graph to an undirected graph is a bit tricky. (After all, any path in an undirected graph can be reversed, and it is still a path, but this is not true for directed graphs.) The idea is to split each vertex of G into three vertices. The first vertex (entry) is used to feed in all incoming edges, the last one (exit) is used to feed out all outgoing edges. The job of the middle vertex is to force any path that comes into the entry vertex to continue through to the exit vertex.

Given a directed graph $G = (V, E)$, the reduction replaces each vertex $u \in V$ with three vertices u (entry), u' (middle), and u'' (exit). We create undirected edges (u, u') and (u', u''). Also, for each directed edge $(u, v) \in E$, we create the undirected edge (v'', u), from v's exit vertex to u's entry vertex.. Let G' be the resulting graph (see Fig. 2(b)).

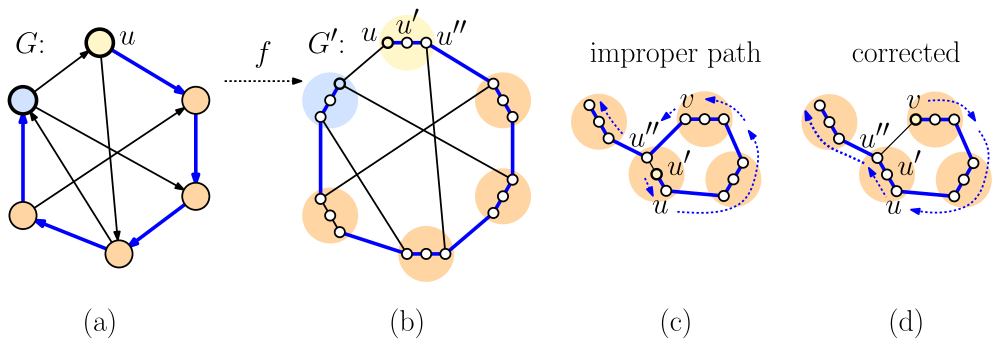


Figure 2: DHP to HP reduction and dealing with improper paths.

It is not hard to see that if G has a Hamiltonian path, then G' will have one as well, by tracing each triple in the proper order (u, u', u''). However, the converse is not at all obvious. Could the path in G' could start at a middle vertex u', for example, which does not correspond to a path in G? That would be a problem! We'll show that if this happens, we can reorient the path so it will start at an entry vertex and end at an exit vertex and visit all vertices along the way.

Claim: The digraph G has a Hamiltonian path if and only if the undirected graph G' has a Hamiltonian path.

Proof: ($\Rightarrow$) If G has a directed Hamiltonian path $\langle u_1, \ldots, u_n \rangle$, we assert that G' has the Hamiltonian path by replacing each vertex u_i with the triple u_i, u_i', u_i'' (see Fig. 2(b)). Since this is Hamiltonian path in G, the edge (u_{i-1}, u_i) is in G, which implies that the edge (u''_{i-1}, u_i) is in G'. By construction the edges (u_i, u_i') and (u_i', u_i'') are in G'. Therefore, this a Hamiltonian path in G'.

($\Leftarrow$) Suppose that G' has a Hamiltonian path. We first claim that we may assume that such a path starts at some vertex u and ends at some vertex v''. If not, we say that the path is *improper*. One way that a path may be improper is that it starts at a middle vertex u'. We will assume that the next vertex on the path is the entry vertex u (see Fig. 2(c)), since a symmetrical argument applies if the path goes next to u''. Since it is Hamiltonian, the path must eventually return to the exit vertex u''. By the nature of G', the vertex v preceding u'' must be an entry vertex. We reverse the path so it starts at v, then goes to u then to u' and u''. After that it follows the original path (see Fig. 2(d)). If the path ends at a middle vertex, a similar correction can be performed.

After this correction, the path starts either at an entry or exit vertex. If it starts at an exit vertex, reverse the entire path, so it starts at an entry vertex. Henceforth, the path must follow the proper structure, entering each vertex triple at the entry vertex and leaving at an exit vertex.

Now that the path is proper, it is easy to see that it corresponds to a valid Hamiltonian path in G, since each edge between two vertices leaves on an exit vertex u'' and enters on an entry vertex v, but this implies that the directed edge (u, v) is part of the original graph.

By the way, you could make the correctness proof simpler by modifying the transformation. We can add a source vertex s to G, which is joined by directed edges from s into to all the vertices of G and a sink vertex t, which is joined by directed edges from all the vertices of G into t. This does not change whether G has a Hamiltonian path, but it simplifies the proof because it is easy to see that any Hamiltonian path in G' must have one endpoint at the entry vertex s and the other at the exit vertex t'', since these vertices have degree 1. The only way the path could be improper is if it starts at t'' and ends at s, in which case we simply reverse it.

(b) What do Hamiltonian paths have in common with spanning trees? Well, a Hamiltonian path is just a spanning tree in which each vertex has degree at most two. How do we make this look like a spanning tree of degree at most three? We just add an extra vertex to each vertex in the original graph, thus boosting the degree of each vertex by one. Observe that these new vertices will not be connected to anything else, so they cannot be used to create a Hamiltonian path where none existed.

Here is the formal reduction. Given an undirected graph $G = (V, E)$ for the HP problem, for each vertex $u \in V$, we create a new vertex u' and add the undirected edge (u, u') (see Fig. 3(a)). Let G' denote the resulting graph. The following lemma shows that this is correct.

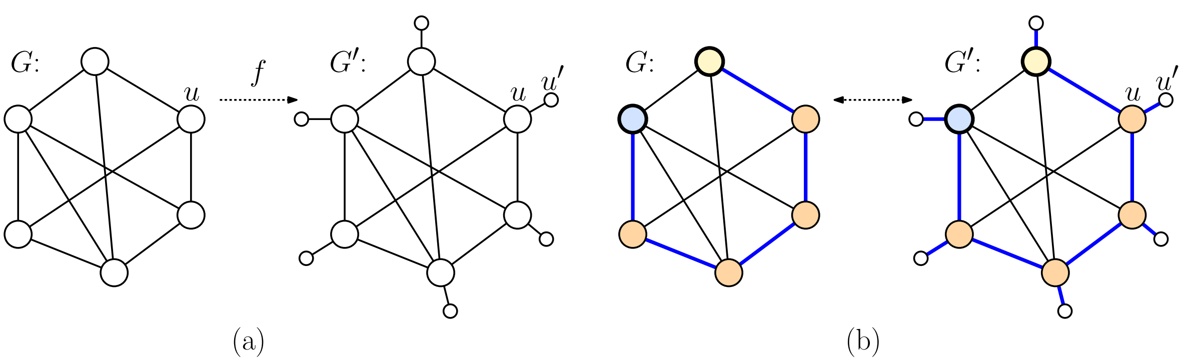


Figure 3: HP to D3ST reduction and correctness.

Claim: G has a Hamiltonian path if and only if G' has a degree-3 spanning tree.

Proof: (⇒) The Hamiltonian path forms a spanning tree of degree at most two in G. Adding the additional edge (u, u') to each vertex creates a spanning tree of degree at most three (see Fig. 3(b)).

(⇐) Observe that each of the newly added edges (u, u') must be in any spanning tree, because this is the only edge incident to u'. Removing these edges decreases the degree of all the remaining vertices by one. These vertices are all from the original graph. Thus, we are left with a spanning tree of degree at most two. Such a spanning tree is a Hamiltonian path.

Solution 2:

B3C $\in$ NP: The certificate is a labeling of the vertices with three colors, say, $\{1, 2, 3\}$. In polynomial time we can check that the each pair of adjacent vertices have different colors, and we can count the number of vertices of each color and verify that each color is used $|V|/3$ times. If so, we accept and otherwise we reject.

B3C $\in$ NP-Hard: We show that the standard 3-coloring problem (3Col) is reducible to it (3Col $\leq_P$ B3C). Let $G = (V, E)$ be an instance of 3Col. We want to produce an equivalent instance of B3C. The trick is to add vertices whose colors can be assigned arbitrarily in order to guarantee that the sizes of the color classes is balanced. However, these new vertices should not affect the colorability of the original graph.

Let $n = |V|$. Let G' consist of a copy of G together with $2n$ additional vertices with no edges attached to them. Observe that G' has $3n$ vertices exactly. Each of these additional vertices is isolated in the sense that it is not adjacent to any other vertex. Clearly G' can be produced from G in polynomial time.

We claim that G is 3-colorable if and only if G' can be 3-colored with colors each occurring equally often.

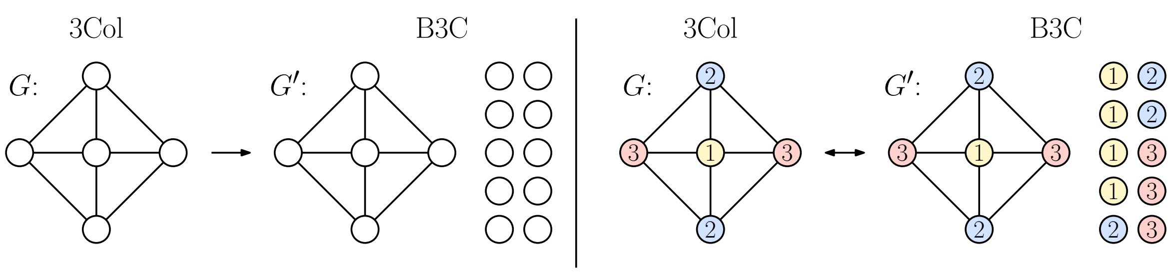


Figure 4: 3Col to B3C reduction and correctness.

- ($\Rightarrow$) Suppose that G is 3-colorable. Let k'_1, k'_2, k'_3 be the number of times each of the colors appears. We have $k'_1 + k'_2 + k'_3 = n$. Because they are isolated, we can color $n - k'_i$ of the isolated vertices with color i. The total number of vertices with color i is thus $k'_i + (n - k'_i) = n$. Since G' has $3n$ vertices, each color is used equally often.
- ($\Leftarrow$) Suppose that G' can be 3-colored with balanced color groups. Then, if we discard the $2n$ newly added vertices, this is a coloring for the remaining graph, namely G.

Solution 3:

(a) To show that kWP is in NP, it suffices to provide a certificate, which will allow us to determine that the graph contains both a k-clique and a k-element independent set. The certificate consists of the two subsets of vertices, V' and V'', each of size k. We then test that for each $(u, v) \in V'$ there is an edge between them (implying that V' is a clique), and for each $(u, v) \in V''$ there is no edge between them (implying that V'' is an independent set).

(b) A k-weird graph must have both a k-clique and a k-independent set. For the reduction, we'll just give the graph a free k-independent set by adding k isolated vertices. Now, it automatically satisfies the independent-set condition, and weirdness relies entirely on the clique property along.

We will show that the clique problem is polynomially reducible to kWP, which implies that kWP is NP-hard. Given a graph G and integer k for the clique problem, we add k new isolated vertices to G (see Fig. 5). Let G' denote the resulting graph and set $k' = k$. Output (G', k'). Clearly, this can be done in polynomial time (and in fact, $O(n + m)$ time).

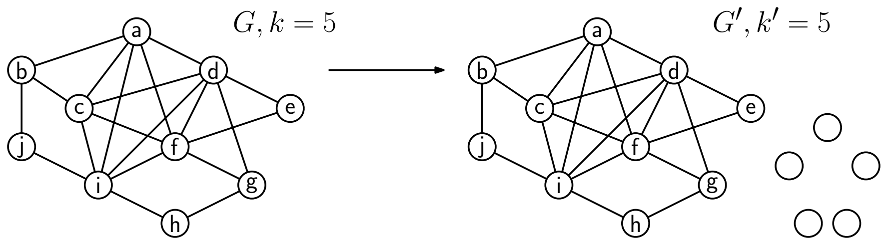


Figure 5: Reducing Clique to kWP.

To establish the correctness of this reduction, we show that G has a clique size k if and only if G' is k-weird.

- ($\Rightarrow$) If G has a clique V' of size k, then the same vertices combined with the new isolated vertices forms a k clique and a k independent set for G'. Therefore, G' is k'-weird, as desired.
- ($\Leftarrow$) If G' is k'-weird, then it contains both a clique V' and an independent set V'', both of size k'. Since the newly added vertices of G' are isolated all of the vertices of V' come from the original graph G. Therefore, G has a clique of size $k' = k$, as desired.

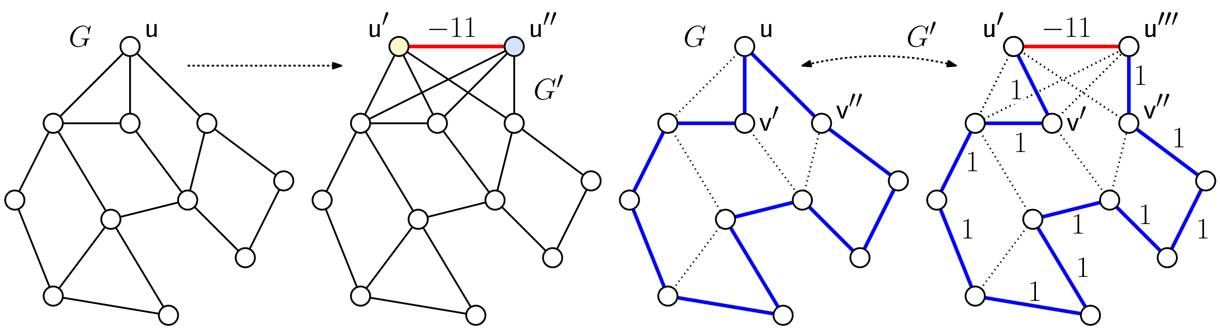
Solution 4:

(a) To show that ZC is in NP, it suffices to provide a certificate, which will allow us to determine that there is a cycle of total weight zero. The certificate consists of the sequence of vertices that forms the cycle. We can traverse this cycle through the graph in polynomial time to determine that (1) it is a valid cycle (it contains at least three vertices and each consecutive pair of vertices are adjacent), (2) it is simple (no vertex is repeated), and (3) the sum of its edge weights is zero.

(b) The zero cycle problem is similar to the Hamiltonian cycle, since the cycle cannot revisit vertices, but how do we use the zero weight condition to enforce the condition that it visits all the vertices. Observe that if we assigned a weight of 1 to every edge, then a cycle is Hamiltonian if and only if its total weight is n, the total number of vertices. In order to force the count to be zero, we just create an edge of weight $-n$, and force the Hamiltonian cycle to use this edge. We'll use a trick similar to the one we used in Problem 1(a). We will split a vertex into two and force the path to travel along an edge between these two vertices. We'll give this edge the weight of $-n$.

We will show that Hamiltonian cycle (HC) is polynomially reducible to ZC, which implies that ZC is NP-hard. Let G be the graph for the HC problem, and let n denote the number of vertices of G. Here is the intuition. Suppose that we can identify an edge e of G that *must* be in the Hamiltonian cycle. We will set this edge's weight to $-n$, and set all the other edge weights to 1. Now, if there is a Zero Cycle in the resulting weighted graph, it must use the only edge of negative weight. Since all the other edges are of weight 1, we need to add n of them to complete the zero cycle. However, (because the zero cycle must be a simple path) such a path must visit all the vertices of G, and hence it is a Hamiltonian cycle.

There are a few ways to do this. Here is one based on local replacement. Consider a graph G for HC (see Fig. 6(a)). Take any vertex u of G. Replace u with two vertices u' and u''. Each is joined to all of u's former neighbors. Add edge (u', u'') and set its weight to $-n$. The remaining edges of G are unchanged and all are assigned weight 1. The resulting weighted graph G' (see Fig. 6(b)) is the output of the reduction. Clearly, this can be done in polynomial time. (More precisely, in $O(n + m)$ time.)



(all other edges weight 1) (Hamitonian cycle) (zero-weight cycle)

(a) (b) (c) (d)

Figure 6: Solution to Problem 4.

To establish the correctness of the reduction, we show that G has a Hamiltonian cycle if and only if G' has a nontrivial simple cycle of weight zero.

- ($\Rightarrow$) Suppose that G has a Hamiltonian cycle. Since this cycle passes through all vertices, it passes through u and two of its neighbors, call them v' and v'' (see Fig. 6(c)). Consider the cycle in G' the results by replacing the triple $\langle v', u, v'' \rangle$ on this cycle with $\langle v', u', u'', v'' \rangle$ (see Fig. 6(d)). By our construction, all these edges exist in G, and so the result is a simple cycle. The Hamiltonian cycle is simple and visits all n vertices of G, and so it contains n edges. Therefore, the sum of the associated edges of G' is $+n$. Adding this to the edge (u', u'') results in a cycle of total weight zero, as desired.
- ($\Leftarrow$) Suppose that G' has a nontrivial zero-weight cycle. Since all the other edges have positive weight, in order to achieve a total weight of zero it must contain at least one

negative weight edge, and the only such edge is (u', u'') of weight $-n$. Since this is a simple cycle, the preceding and following vertices must be two distinct vertices neighbors v' and v'' from the set of u's original neighbors in G. In order to achieve a weight of zero, the cycle must contain n other edges, meaning that, in addition to u' and u'', it visits $n - 1$ vertices of G'. If we replace u' and u'' with u, it follows that the result is a simple cycle in G that visits all n vertices of G, that is, a Hamiltonian cycle, as desired.