CMSC 451: Lecture 17
NP-Completeness: Clique, Vertex Cover, and Dominating Set

Recap: Last time we gave a reduction from 3SAT (satisfiability of boolean formulas in 3-CNF form) to IS (independent set in graphs). Today we give a few more examples of reductions. Recall that to show that a decision problem (language) L is NP-complete we need to show:

(i) $L \in$ NP. (That is, given an input and an appropriate certificate, we can guess the solution and verify whether the input is in the language), and

(ii) L is NP-hard, which we can show by giving a reduction from some known NP-complete problem L' to L, that is, $L' \leq_P L$. (That is, there is a polynomial time function that transforms an instance L' into an equivalent instance of L for the other problem).

Some Easy Reductions: Next, let us consider some closely related NP-complete problems:

Clique (CLIQUE): The *clique problem* is: given an undirected graph $G = (V, E)$ and an integer k, does G have a subset V' of k vertices such that for each distinct $u, v \in V'$, $(u, v) \in E$. In other words, does G have a k vertex subset whose induced subgraph is complete? (See Fig. 1(a).)

Vertex Cover (VC): A *vertex cover* in an undirected graph $G = (V, E)$ is a subset of vertices $V' \subseteq V$ such that every edge in G has at least one endpoint in V'. The *vertex cover problem* (VC) is: given an undirected graph G and an integer k, does G have a vertex cover of size k? (See Fig. 1(b).)

Dominating Set (DS): A *dominating set* in a graph $G = (V, E)$ is a subset of vertices V' such that every vertex in the graph is either in V' or is adjacent to some vertex in V'. The *dominating set problem* (DS) is: given a graph $G = (V, E)$ and an integer k, does G have a dominating set of size k? (See Fig. 1(c).)

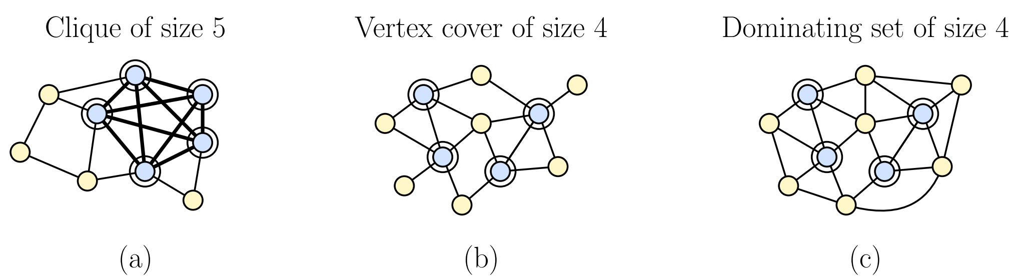

Fig. 1: Clique, Vertex Cover and Dominating Set.

Don't confuse the clique (CLIQUE) problem with the clique-cover (CC) problem that we discussed in an earlier lecture. The clique problem seeks to find a single clique of size k, and the clique-cover problem seeks to partition the vertices into k groups, each of which is a clique.

We have discussed the facts that cliques are of interest in applications dealing with clustering. The vertex cover problem arises in various servicing applications. For example, you have a compute network and a program that checks the integrity of the communication links. To save the space of installing the program on every computer in the network, it suffices to install it on all the computers forming a vertex cover. From these nodes all the links can be tested. Dominating set is useful in facility location problems. For example, suppose we want to select where to place a set of fire stations such that every house in the city is within two minutes of the nearest fire station. We create a graph in which two locations are adjacent if they are within two minutes of each other. A minimum sized dominating set will be a minimum set of locations such that every other location is reachable within two minutes from one of these sites.

The CLIQUE problem is obviously closely related to the independent set problem (IS): Given a graph G does it have a k vertex subset that is completely *disconnected*. It is not quite as clear that the vertex cover problem is related. However, the following lemma makes this connection clear as well (see Fig. 2). Given a graph G, recall that $\overline{G}$ is the *complement graph* where edges and non-edges are reverse. Also, recall that $A \setminus B$ denotes set resulting by removing the elements of B from A.

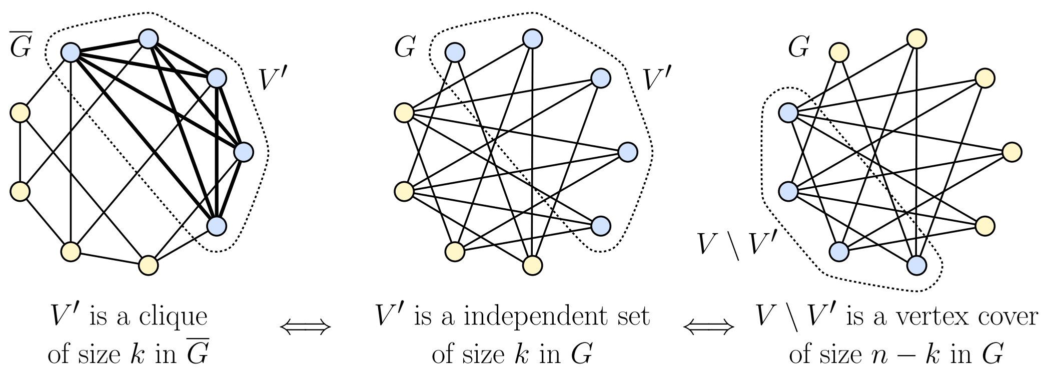

Fig. 2: Clique, Independent set, and Vertex Cover.

Lemma: Given an undirected graph $G = (V, E)$ with n vertices and a subset $V' \subseteq V$ of size k. The following are equivalent:

(i) V' is a clique of size k for the complement, $\overline{G}$

(ii) V' is an independent set of size k for G

(iii) $V \setminus V'$ is a vertex cover of size $n - k$ for G, (where $n = |V|$)

Proof:

- $(i) \Rightarrow (ii)$: If V' is a clique for $\overline{G}$, then for each $u, v \in V'$, (u, v) is an edge of $\overline{G}$ implying that (u, v) is not an edge of G, implying that V' is an independent set for G.
- $(ii) \Rightarrow (iii)$: If V' is an independent set for G, then for each $u, v \in V'$, (u, v) is not an edge of G, implying that every edge in G is incident to a vertex in $V \setminus V'$, implying that $V \setminus V'$ is a vertex cover for G.
- $(iii) \Rightarrow (i)$: If $V \setminus V'$ is a vertex cover for G, then for any $u, v \in V'$ there is no edge (u, v) in G, implying that there is an edge (u, v) in $\overline{G}$, implying that V' is a clique in $\overline{G}$.

Thus, if we had an algorithm for solving any one of these problems, we could easily translate it into an algorithm for the others. In particular, we have the following.

Theorem: CLIQUE is NP-complete.

Proof:

- CLIQUE $\in$ NP: Consider an instance (G, k) for CLIQUE. The certificate consists of k vertices of G forming the set V'. We can check that all pairs of vertices in V' are adjacent (e.g., by inspection of $O(k^2) = O(n^2)$ entries of the adjacency matrix). If so, the verification succeeds and we accept, and otherwise the verification fails and we reject.
- IS $\leq_P$ CLIQUE: We want to show that given an instance of the IS problem (G, k), we can produce an equivalent instance of the CLIQUE problem in polynomial time. The reduction function f inputs G and k, and outputs the pair $(\overline{G}, k)$. Clearly this can be done in polynomial time. By the above lemma, this instance is equivalent.

Theorem: VC is NP-complete.

Proof:

- VC $\in$ NP: Consider an instance (G, k) for VC. The certificate consists of k vertices of G forming the set V'. In $O(m) = O(n^2)$ time we can check that every edge in G has at least one endpoint in V'. If so, the verification succeeds and we accept, and otherwise the verification fails and we reject.
- IS $\leq_P$ VC: We want to show that given an instance of the IS problem (G, k), we can produce an equivalent instance of the VC problem in polynomial time. The reduction function f inputs G and k, computes the number of vertices, n, and then outputs $(G, n - k)$. Clearly this can be done in polynomial time. By the above lemma, these instances are equivalent.

We reiterate that in each of the above reductions, the reduction function merely translates similar elements between the two problems. It does not know whether G has an independent set or not. Even if it did, it does not know which vertices are in the independent set.

Dominating Set: In spite of the superficial similarity to Vertex Cover, Dominating Set is a bit trickier to show NP-completeness. As usual the proof has two parts. First, we show that DS $\in$ NP (see below). The trickier part is showing the some known NP-complete problem is reducible to DS. We will show that VC $\leq_P$ DS. That is, we want to show that there is a polynomial time function, which given an instance (G, k) for VC, produces an instance (G', k') for DS, such that G has a vertex cover of size k if and only if G' has a dominating set of size k'.

How to we translate between these problems? The key difference is the covering condition.

- VC: Every edge is incident to a vertex in V'.
- DS: Every vertex is either in V' or is adjacent to a vertex in V'.

Thus the translation must somehow map the notion of "incident edge" to "adjacent vertex". Because incidence is a property of edges, and adjacency is a property of vertices, this suggests that the reduction function maps edges of G into vertices in G', such that an incident edge in G is mapped to an adjacent vertex in G'.

This inspires the following idea. We will insert a vertex into the middle of each edge of the graph. In other words, for each edge (u, v), we will create a new *mid-edge vertex*, called w_{uv}, and replace the edge (u, v) with the two edges (u, w_{uv}) and (v, w_{uv}) (see Fig. 3). The fact that u was incident to edge (u, v) has now been replaced with the fact that u is adjacent to the corresponding vertex w_{uv}. We still need to dominate the neighbor v. To do this, we will leave the edge (u, v) in the graph as well. Let G' be the resulting graph.

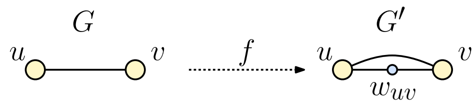

Fig. 3: Gadget for the VC $\leq_P$ DS reduction.

This is still not quite correct though. Define an *isolated vertex* to be one that is incident to no edges. If u is isolated it can only be dominated if it is included in the dominating set. Since it is not incident to any edges, it does not need to be in the vertex cover. Let V_I denote the isolated vertices in G, and let n_I denote the number of isolated vertices. The number of vertices to request for the dominating set will be $k' = k + n_I$. Okay, we are now ready to state the result and prove it.

Theorem: DS is NP-complete.

Proof:

- DS $\in$ NP: Given an instance (G, k) for DS, we guess the certificate, which consists of the k vertices that will form the dominating set. We then verify that these vertices form a dominating set, by checking that every vertex of G is either in this set or is adjacent to a vertex in this set. If so, we output "yes" and otherwise "no". (Again, if G has a dominating set of size k, one of these guesses will work, and we correctly classify G as having a dominating set of size k. Otherwise, all fail and we classify G as not having such a dominating set.)
- VC $\leq_P$ DS: We want to show that given an instance of the VC problem (G, k), we can produce an equivalent instance of the DS problem in polynomial time. We create a graph G' as follows. Initially $G' = G$. For each edge (u, v) in G we create a new vertex w_{uv} in G' and add edges (u, w_{uv}) and (v, w_{uv}) in G'. Let I denote the number of isolated vertices and set $k' = k + n_I$. Output (G', k'). This reduction illustrated in Fig. 4. Note that every step can be performed in polynomial time.
  To establish the correctness of the reduction, we need to show that G has a vertex cover of size k if and only if G' has a dominating set of size k'.
  ($\Rightarrow$) First we argue that if V' is a vertex cover for G, then $V'' = V' \cup V_I$ is a dominating set for G'. Observe that

  $$|V''| \;=\; |V' \cup V_I| \;\leq\; k + n_I \;=\; k'.$$

  Note that $|V' \cup V_I|$ might be of size less than $k + n_I$, if there are any isolated vertices in V'. If so, we can add any vertices we like to make the size equal to k' (see Fig. 5).

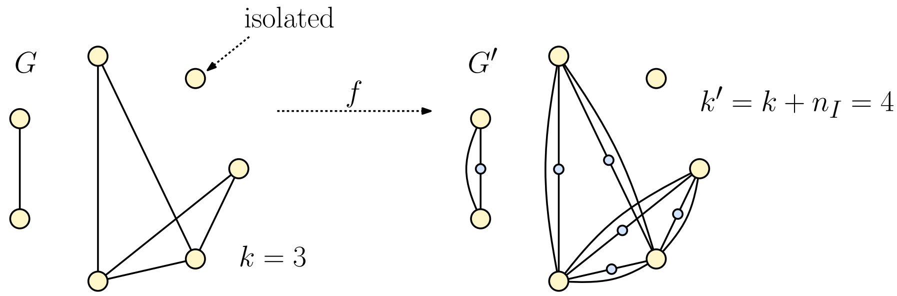

Fig. 4: Dominating set reduction with $k = 3$ and one isolated vertex.

To see that V'' is a dominating set, first observe that all the isolated vertices are in V'' and so they are dominated. Second, each of the mid-edge vertices w_{uv} in G' corresponds to an edge (u, v) in G implying that either u or v is in the vertex cover V'. Thus w_{uv} is dominated by the same vertex in V'' Finally, each of the nonisolated original vertices v is incident to at least one edge in G, and hence either it is in V' or else all of its neighbors are in V'. In either case, v is either in V'' or adjacent to a vertex in V'' (see Fig. 5).

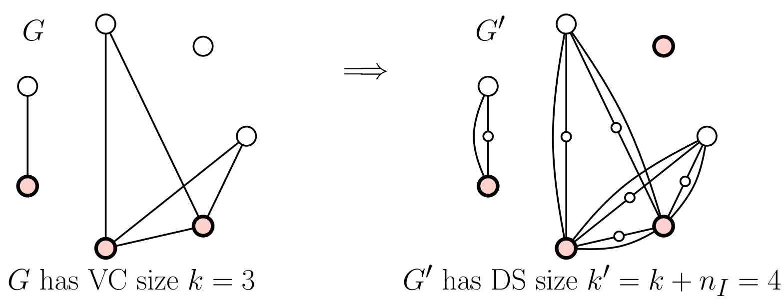

Fig. 5: $\Rightarrow$ part of the correctness of the VC to DS reduction (where $k = 3$ and $I = 1$).

($\Leftarrow$) Conversely, we claim that if G' has a dominating set V'' of size $k' = k + n_I$ then G has a vertex cover V' of size k. Note that all n_I isolated vertices of G' must be in the dominating set (see Fig. 6). First, let $V''' = V'' \setminus V_I$ be the remaining k vertices. We might try to claim something like: V''' is a vertex cover for G. But this will not necessarily work, because V''' may have vertices that are not part of the original graph G.

However, we claim that we never need to use any of the newly created mid-edge vertices in V'''. In particular, if some vertex $w_{uv} \in V'''$, then modify V''' by replacing w_{uv} with u. (We could have just as easily replaced it with v.) Observe that the vertex w_{uv} is adjacent to only u and v, so it dominates itself and these other two vertices. By using u instead, we still dominate u, v, and w_{uv} (because u has edges going to v and w_{uv}). Thus by replacing $w_{u,v}$ with u we dominate the same vertices (and potentially more). Let V' denote the resulting set after this modification. (This is shown in Fig 6.)

We claim that V' is a vertex cover for G. If, to the contrary there were an edge (u, v)

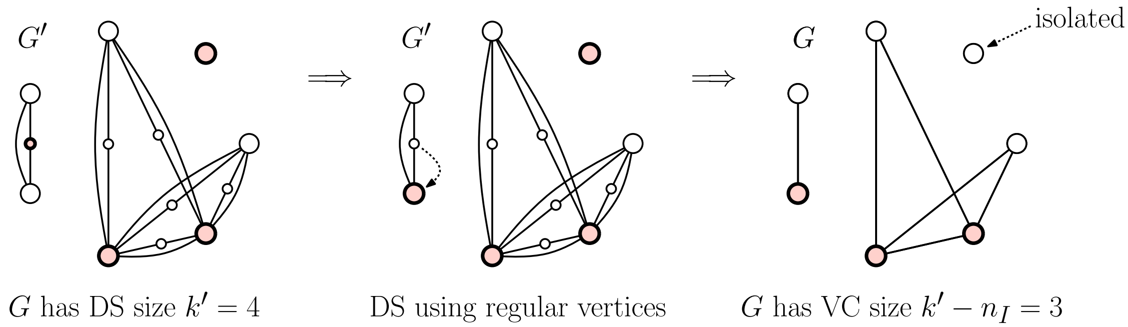

Fig. 6: $\Leftarrow$ part of the correctness of the VC to DS reduction (where $k = 3$ and $I = 1$).

of G that was not covered (neither u nor v was in V') then the mid-edge vertex w_{uv} would not be adjacent to any vertex of V'' in G', contradicting the hypothesis that V'' was a dominating set for G'.

Whew! In conclusion, DS is NP-complete.

Summary: In this lecture we expanded our set of known NP-complete problems to include Clique, Vertex Cover, and Dominating Set.