

CMSC 451 - Algorithm Design

Lecture 14 - Network Flow - Circulations + Applications

Circulation with Demands:

Transportation:

- Network (where commodities flow)
- Edge capacities
- Supply nodes - generate flow
- Demand nodes - consume flow

Decision problem:

- Can we manage all the flow from suppliers to consumers?

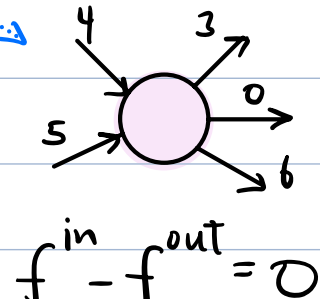
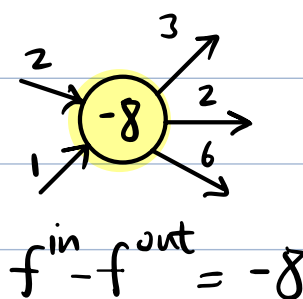
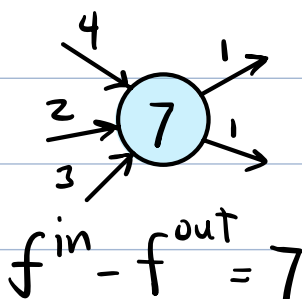
Supply + Demand: Given digraph $G=(V,E)$

- Each node $v \in V$ has demand $d(v)$:

demand: $d(v) > 0$ (d_v net incoming)

supply: $d(v) < 0$ ($-d_v$ net outgoing)

balanced: $d(v) = 0$ (incoming = outgoing)



Circulation: Given a directed graph $G=(V,E)$ with edge capacities $c(u,v) > 0$ and vertex demands $d(v)$, a circulation is a function $f: E \rightarrow \mathbb{R}$ such that:

Capacity constraint: $0 \leq f(u,v) \leq c(u,v), \forall (u,v) \in E$

Supply / Demand constraint:

$$f^{\text{in}}(v) - f^{\text{out}}(v) = d(v), \forall v \in V$$

Recall:

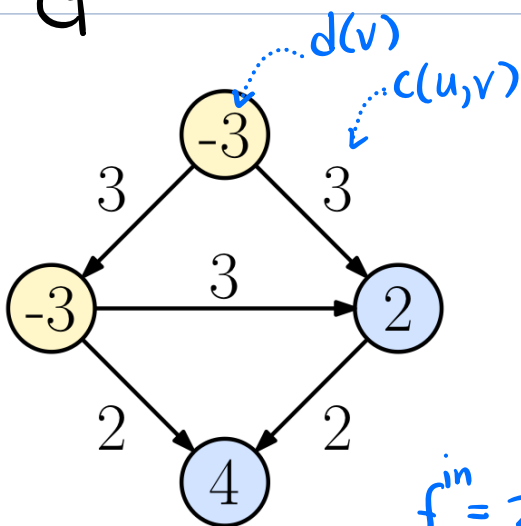
$$f^{\text{in}}(v) = \sum_{(u,v) \in E} f(u,v)$$

$$f^{\text{out}}(v) = \sum_{(v,w) \in E} f(v,w)$$

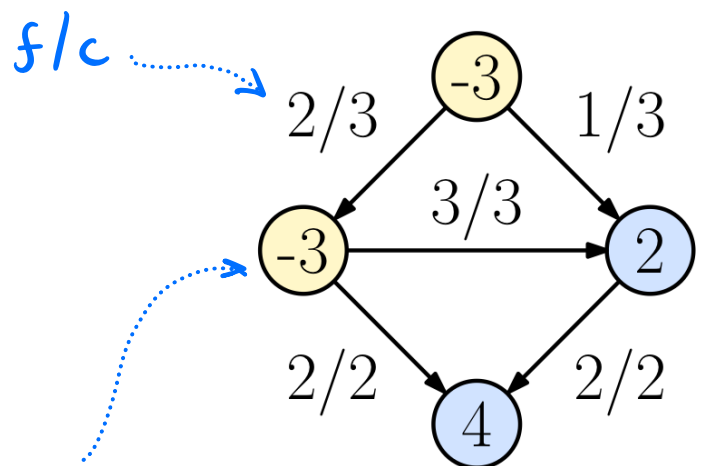
⑤ supply node

④ demand node

G :



A circulation:



$$f^{\text{in}} = 2$$

$$f^{\text{out}} = 3 + 2 = 5$$

$$f^{\text{in}} - f^{\text{out}} = 2 - 5 = -3 = d(v)$$

Define: $S = \{v \mid d(v) < 0\}$ Supply nodes
 $T = \{v \mid d(v) > 0\}$ Demand nodes

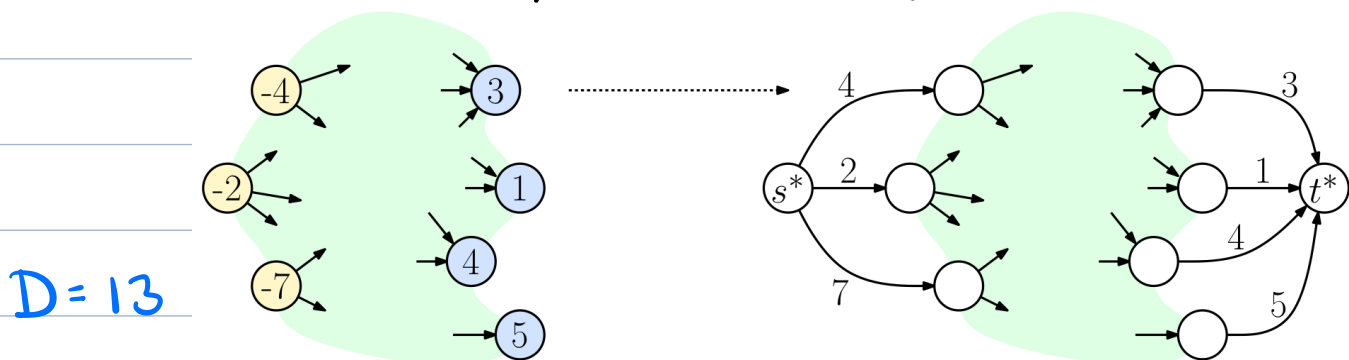
Clearly: $\sum_{v \in T} d(v) = - \sum_{v \in S} d(v)$

demand = supply
 (or no feasible circulation exists)

Define: Total demand: $\sum_{v \in T} d(v)$ denoted $D(G)$

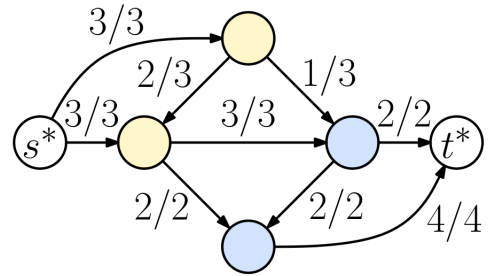
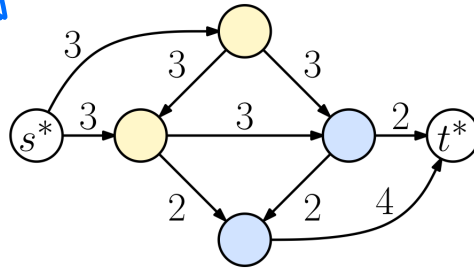
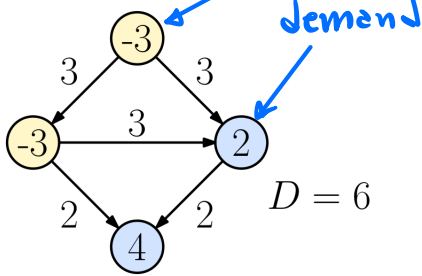
Solving Circulation by Network Flow

- Treat supply nodes as sources
- Treat demand nodes as sinks
- Add super-source + super-sink to connect



- Create network $G' = (V', E')$, $V' \leftarrow V$, $E' \leftarrow E$
- Add s^* + t^* to V'
- For $v \in V$: Add (s^*, v) capacity = $-d(v)$ if $v \in S$
 Add (v, t^*) capacity = $d(v)$ if $v \in T$

Example:



Claim: G has a feasible circulation iff G' has a flow of value $D(G)$

Recall:

$$= \sum_{v \in T} d(v)$$

$$= - \sum_{v \in S} d(v)$$

Proof:

($\Rightarrow$) Let f be a circulation in G .

Create flow f' by:

- saturate all edges out of s^*
- " " " into t^*
- all other edges same as f

This satisfies -

- capacity constraints
- supply-demand constraints

} Exercise

($\Leftarrow$) Let f' be flow of value D in G'

Obs: All edges out of s^* + into t^* must be saturated

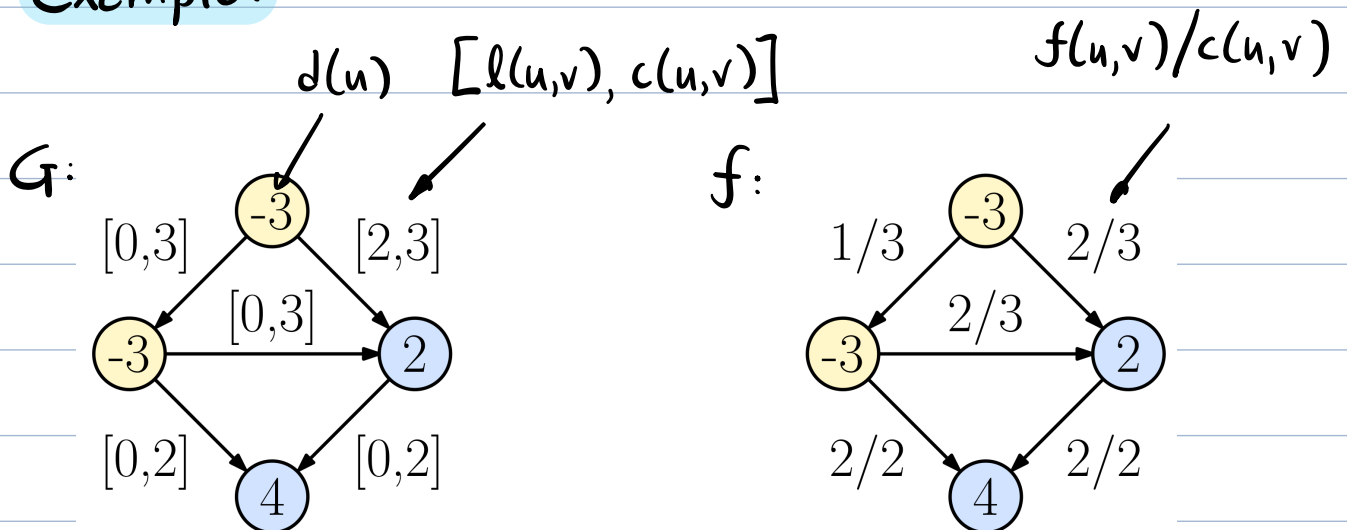
$\Rightarrow$ All supply/demands satisfied

} Exercise

Circulations with Upper + Lower Capacity Bounds

- Sometimes, we want to apply both upper + lower bounds on flow amounts
- Lower bound $\Rightarrow$ guarantees minimum utilization
- For each edge $(u,v) \in E$, let:
 - $c(u,v)$ = upper bound on flow
 - $l(u,v)$ = lower bound on flow
- New conditions: A flow is a circulation if
 - $l(u,v) \leq f(u,v) \leq c(u,v), \forall (u,v) \in E$
 - $f^{\text{in}}(v) - f^{\text{out}}(v) = d(v), \forall v \in V$

Example:

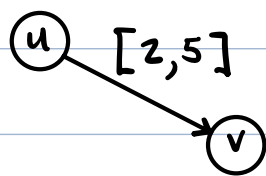


But how can we enforce these lower bounds?

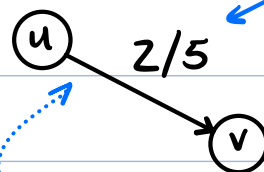
- Can do this by "mucking" with demands!



Step 1: Start by forcing flow of $l(u,v)$ on each edge (u,v)



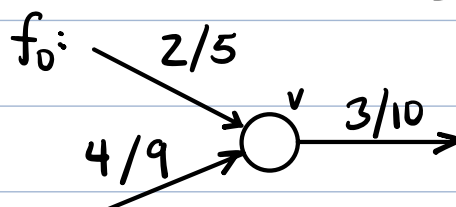
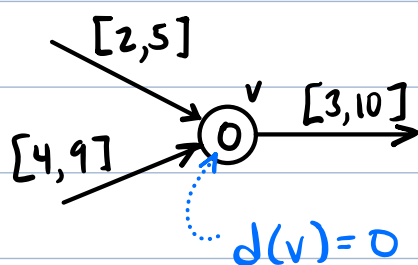
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Force 2 units of flow on (u,v)

Call this f_0 (pseudo-flow)

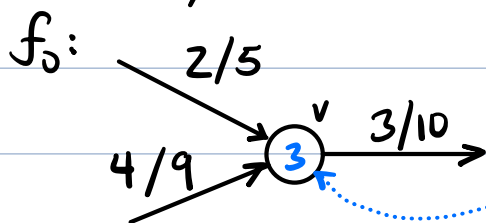
But this forced flow won't generally satisfy vertex demands:



$$f_0^{\text{in}}(v) - f_0^{\text{out}}(v) = (2+4) - 3 = 3 \neq d(v) \text{ :-(}$$

Step 2: Determine excess flow through each vertex

$$\chi(v) = f_0^{\text{in}}(v) - f_0^{\text{out}}(v)$$

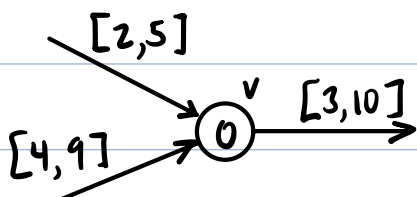


$$f_0^{\text{in}}(v) - f_0^{\text{out}}(v) = (2+4) - 3 = 3 = \chi(v)$$

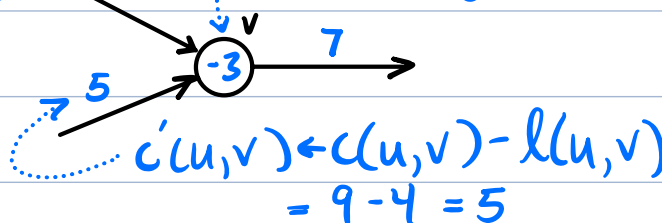
Step 3: Eliminate f_0 by reducing upper capacities + adjusting demands:

$$c'(u,v) \leftarrow c(u,v) - f_0(u,v) = c(u,v) - l(u,v)$$

$$d'(v) \leftarrow d(v) - \chi(v)$$

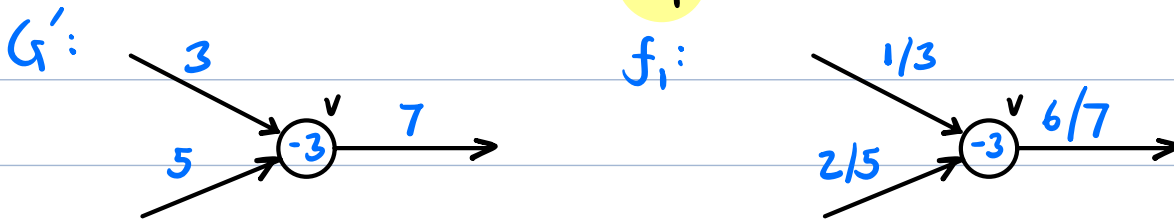


$$d'(v) = d(v) - \chi(v) = 0 - 3 = -3$$

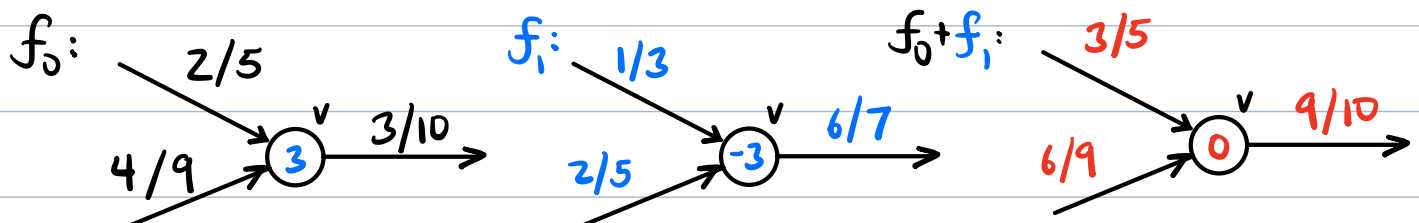


Observe: Lower constraints gone - just upper remain

Step 4: Apply standard circulation algorithm to this modified network G'
→ Let f_1 be result



Step 5: Add back the (eliminated) forced lower flow:
return $f_0 + f_1$



Summary: Given G with $[l, u]$ capacity constraints

- Create pseudo-flow $f_0(u, v) \leftarrow l(u, v)$
- Create network G' (same vertices + edges)
 - $c'(u, v) \leftarrow c(u, v) - l(u, v)$
 - $d'(u, v) \leftarrow d(v) - x(v)$, $x(v) = f_0^{\text{in}}(v) - f_0^{\text{out}}(v)$
- Compute (standard) circulation in $G' \rightarrow f_1$
- If G' has no circulation - report infeasible
else return $f \leftarrow f_0 + f_1$

Correctness:

Lemma: G (with both lower + upper constraints) has a valid circulation iff G' (just upper constraints) has a valid circulation.

See Kleinberg + Tardos for proof

- Intuitively: f_0 handles lower bounds
- d' handles imbalance in demands
- c' limits additional flow above $l(u,v)$ so it doesn't exceed $c(u,v)$
- $f_0 + f_1$ combines the two flows

Application - Survey Design

Setup:

- A company sells k products
- Survey n customers - "How satisfied..."
- Rules:



- ① Only ask about products customer purchased
- ② Customer i : Ask about $\geq c_i^-$ products
and $\leq c_i^+$ products
- ③ Product j : Survey $\geq p_j^-$ customers
and $\leq p_j^+$ customers

Q: Given $C = \{[c_i^-, c_i^+], \dots, [c_n^-, c_n^+]\}$

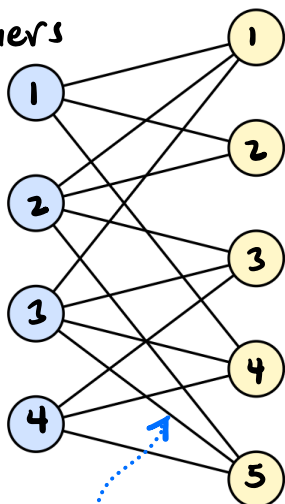
$P = \{[p_i^-, p_i^+], \dots, [p_k^-, p_k^+]\}$

+ purchase information

does there exist a survey satisfying all rules?

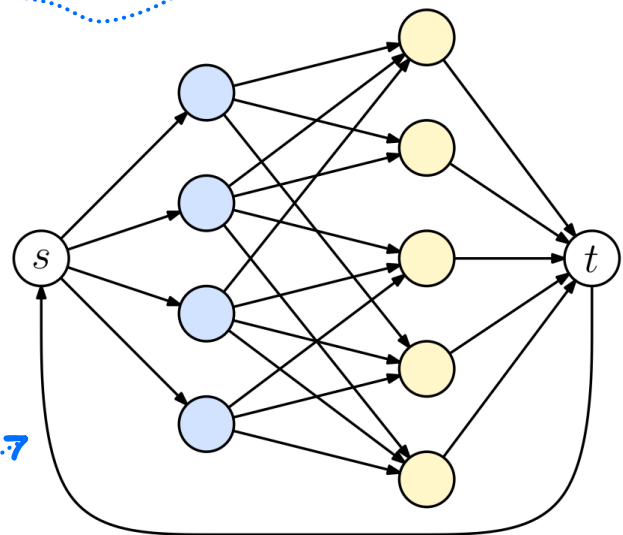
Purchase information - Modeled as bipartite graph

customers



customer 3
purchased
product 5

products



All demands = 0

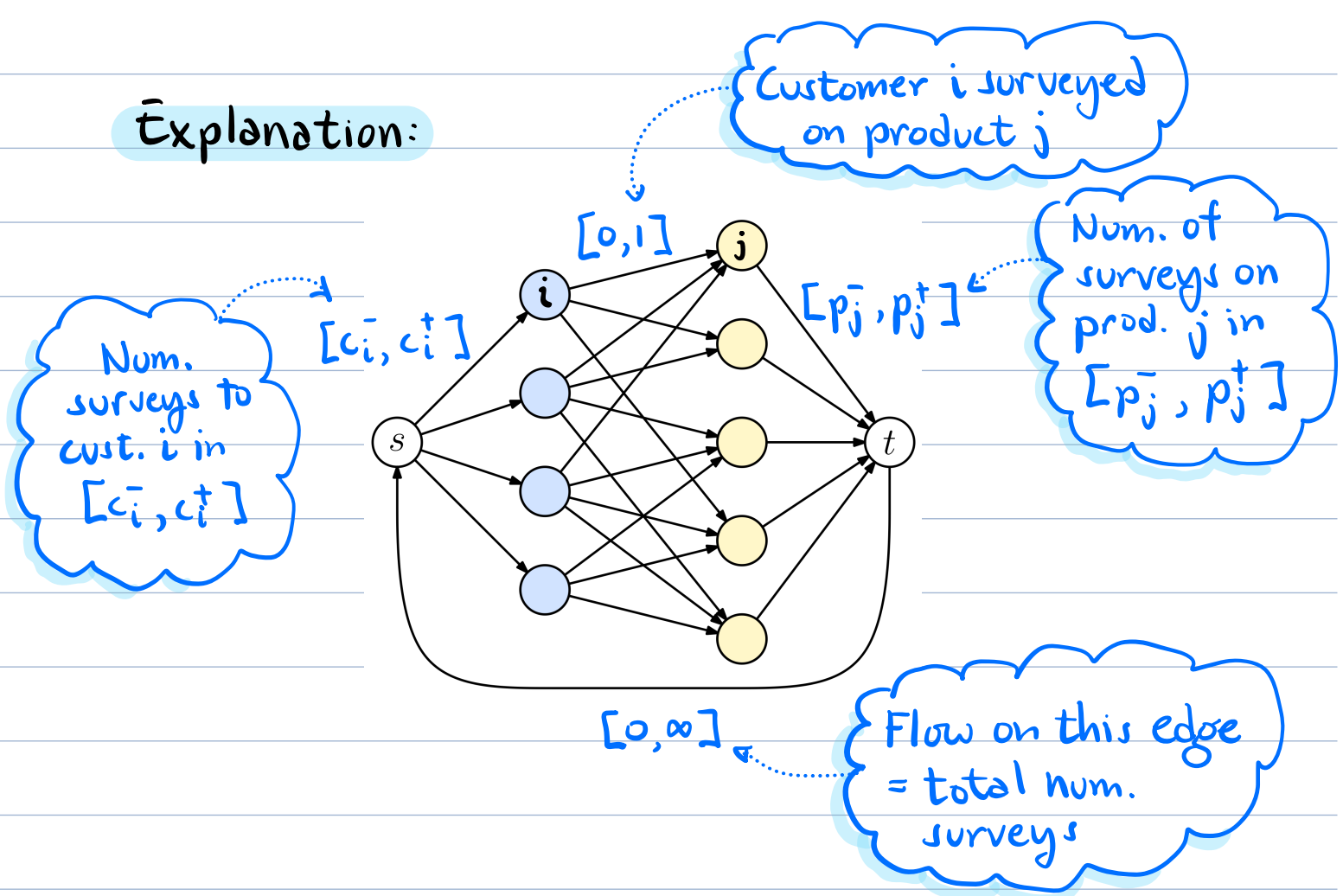
(flow conservation)

- Direct edges from cust. to prod.
- Add vertex s + edges to customers
- Add vertex t + edges from products
- Add edge (t, s)

Capacities:

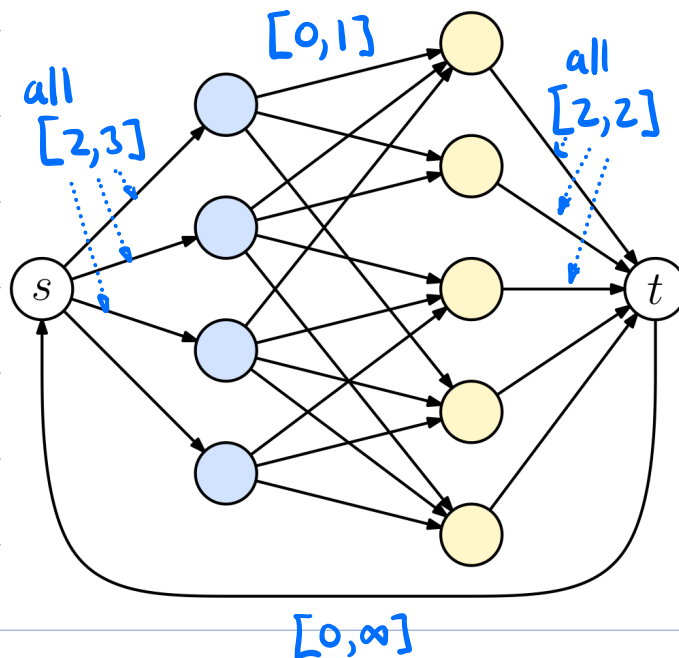
- Customer edges $(s, i): [c_i^-, c_i^+]$
- Product edges $(j, t): [p_j^-, p_j^+]$
- Customer/product edges: $[0, 1]$
- $(t, s): [0, \infty]$

Explanation:

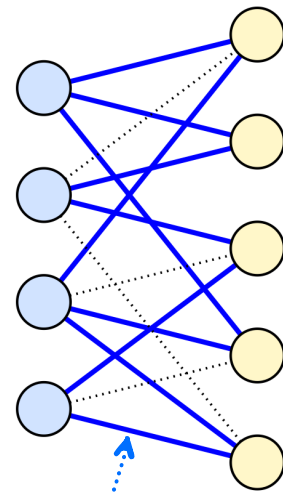
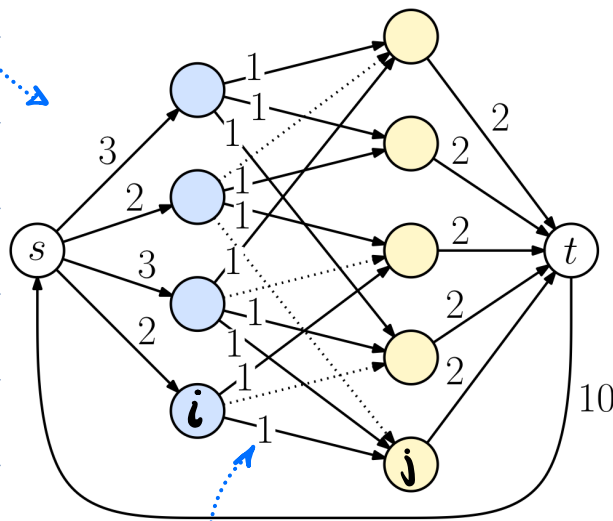


Example:

- For simplicity, set $[c_i^-, c_i^+] = [2, 3]$, for all i
 $[p_j^-, p_j^+] = [2, 2]$, for all j



Compute Circulation - f (integer valued)



if $f(i,j)=1$ survey
customer i about
product j

Is this a faithful implementation of survey rules?

Claim: There exists a valid circulation in G
iff there is a valid survey design

Proof:

($\Rightarrow$) Suppose G has a valid circulation f .

- Survey: Ask cust. i about prod j if $f(i,j)=1$

- We'll show this survey is valid:

① Only survey customers about purchases

$\Leftarrow$ Create edge (i,j) if i purchases j

- ② Num. of surveys for cust i in $[c_i^-, c_i^+]$
 $\Leftarrow$ Since $f(s, i) \in [c_i^-, c_i^+]$
 + flow conservation ($d(i) = 0$)
- ③ Num. of surveys for prod. j in $[p_j^-, p_j^+]$
 $\Leftarrow$ Since $f(j, t) \in [p_j^-, p_j^+]$
 + flow conservation ($d(j) = 0$)
- $\Rightarrow$ This is a valid survey $\checkmark$

($\Leftarrow$) Suppose there is a valid survey.

Construct flow in G :

- $f(i, j) = 1$ if cust. i surveyed on prod j
- $f(s, i) =$ total surveys for cust i
- $f(j, t) =$ total surveys for prod j
- $f(t, s) =$ total num. of surveys

We'll show this is a valid circulation:

- ① Cust. i surveyed about prod. j
 only if purchased $\Rightarrow$ edge $(i, j) \in G$
- ② Cust. i surveyed about $[c_i^-, c_i^+]$ products
 $\Rightarrow f(s, i)$ satisfies capacities
- ③ Prod. j involved in $[p_j^-, p_j^+]$ surveys
 $\Rightarrow f(j, t)$ satisfies capacities

Finally flows-in = flows-out so all demands (zero) are met $\Rightarrow$ valid circulation $\checkmark$

Ugh!



- These proofs are lengthy + rather tedious.
- Important to understand structure
 - Similar to NP-complete reductions

Summary:

- Circulations - vertex demand/supply
 - Reduction to max flow
- Circulations with lower + upper edge capacities
 - Reduction standard circulations
- Application - Survey Design
 - Reduction to circulations w. lower/upper capacities