

CMSC 451 - Algorithm Design

Lecture 17 - NP-Completeness: Clique, Vert. Cov. + Dom. Set

This lecture - 3 easy reductions

Recap: To show L is NP-complete:

(i) $L \in NP$ - it is verifiable

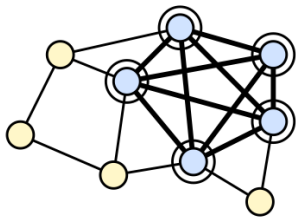
(ii) L is NP-hard - suffices to show $L' \leq_p L$
for some known NP-hard problem L'

Three problems:

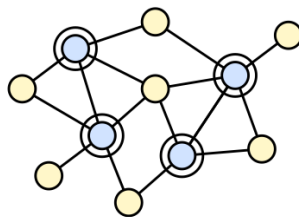
Clique (CLIQUE): Given a graph $G=(V,E)$ + integer k ,
is there a subset $V' \subseteq V$ of size k s.t.

all pairs of vertices in V' are adjacent.

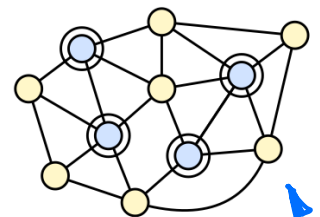
Clique of size 5



Vertex cover size 4



Dominating set size 4



Vertex Cover (VC): Given graph $G=(V,E)$ + integer k ,
is there a subset $V' \subseteq V$ of size k s.t.

every edge of G is incident to some vert. of V'

Dominating Set (DS): Given graph $G=(V,E)$ + integer k , is there a subset $V' \subseteq V$ of size k s.t. every vertex in G is in V' or adjacent to a vertex in V'

Applications:

Clique - Analysis of tightly coupled groups in social networks

Vertex Cover + Dominating Set - Used in applications in facility location (placing service facilities close to clients)

Clique / IS / VC: Closely related.

Recall: $\bar{G}$ = complement graph $(u,v) \in \bar{G}$ iff $(u,v) \notin G$

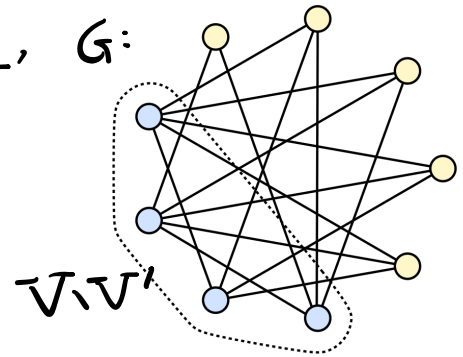
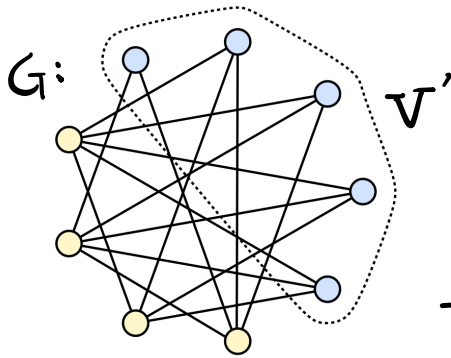
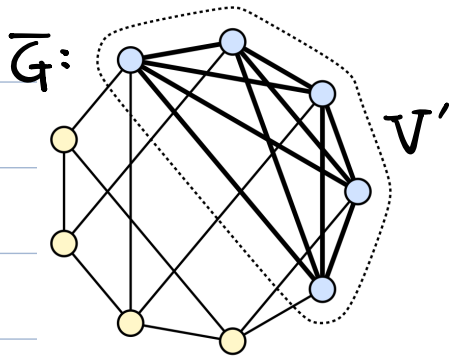
Lemma: Given a graph $G=(V,E)$ with $n=|V|$. Let $V' \subseteq V$ + $k=|V'|$. Then the following are equivalent:

- (i) V' is a clique of size k in $\bar{G}$
- (ii) V' is an indep. set of size k in G
- (iii) $V \setminus V'$ is a vertex cover of size $n-k$ in G

V' is clique in $\bar{G}$

V' is indep set in G

$V \setminus V'$ is VC in G



Proof: (sketch)

(i) $\Leftrightarrow$ (ii): All adjacent in $\bar{G}$

$\Leftrightarrow$ none adjacent in G

(ii) $\Leftrightarrow$ (iii): No edges among V'

$\Leftrightarrow$ all edges touch $V \setminus V'$

Theorem: CLIQUE is NP-complete

Proof: (i) CLIQUE $\in$ NP

Given input $(G=(V,E), k)$

certificate = list of k vertices $\rightarrow V'$

verification:

for each $u, v \in V'$

check that $(u, v) \in E$

if yes for all - accept

else - reject

(ii) $IS \leq_p \text{CLIQUE}$

- Given instance (G, k) for IS
our translation function outputs $(\bar{G}, k)$ for CLIQUE .
- Can complement graph in $O(n^2)$ time
- Correctness:
 G has indep. set of size k
iff
 $\bar{G}$ has CLIQUE of size k
(by prior lemma (i) $\Leftrightarrow$ (ii)) $\square$

Theorem: VC is NP-complete

Proof: (i) $\text{VC} \in \text{NP}$

Given input $(G=(V, E), k)$

certificate = list of k vertices $\rightarrow V'$

verification:

for each edge $(u, v) \in E$

check that $u \in V'$ or $v \in V'$

if yes for all - accept

else - reject

(ii) $IS \leq_p VC$

- Given instance (G, k) for IS
our translation function outputs
 $(G, n-k)$ for VC ($n = |V|$)
- Can compute in $O(n)$ time
- Correctness:
 G has indep. set of size k
iff
 G has vertex cover of size $n-k$
(by prior lemma (ii) $\Leftrightarrow$ (iii)) $\square$

Theorem: DS is NP-complete

- This is trickier!
 - Looks similar to VC (if V' is a vertex cover then it's a dom. set, right?)
- But, translation must be faithful
(if V' is a dom. set it may not be VC)

- Goal: Poly-time function $f: (G, k) \rightarrow (G', k')$ s.t.
 G has VC of size k
iff
 G' has DS of size k'

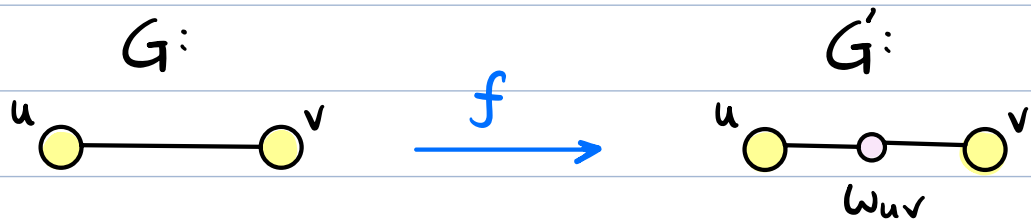
- What are corresponding elements?

VC: Every edge is incident to V'

DS: Every vertex is adjacent (or in) V'

$\Rightarrow$ Need to make edges behave like vertices

Idea 1: Add a vertex within each edge:



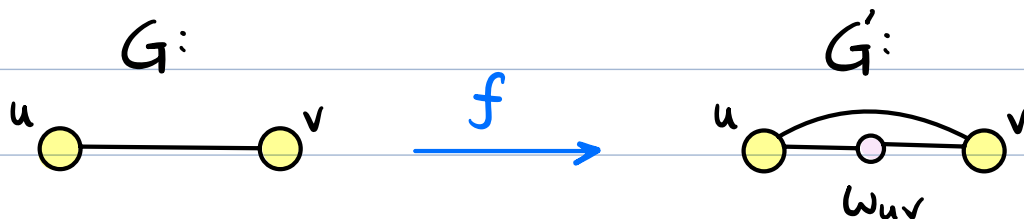
Putting u or v in VC
covers (u, v)

Putting u or v in DS
dominates w_{uv}

$\rightarrow$ But if u in DS, who dominates v ?



Idea 2: Add a vertex within each edge + keep the old edge.



Putting u or v in VC
covers (u, v)

Putting u/v in DS
dominates w_{uv} + v/u

This is almost perfect. Two issues:

- ① Don't want DS to use the mid-edge vertices. How to enforce this?
- ② Who dominates isolated vertices? (no incident edges)

Fixes:

- ① Not a problem. No advantage in using w_{uv} , since u or v dominates at least as many vertices.



② Isolated vertices are trivial to handle:

VC - They never help

DS - They must be included

Let n_I denote no. of isolated vertices in G . Set $k' \leftarrow k + n_I$

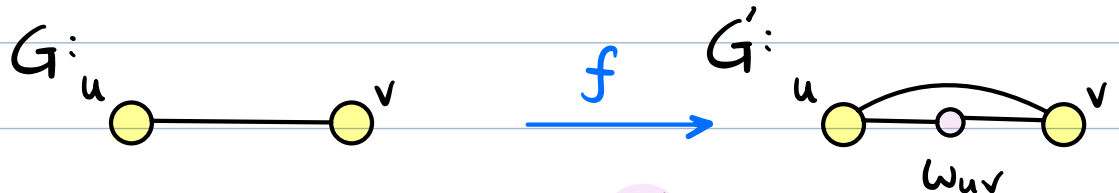
Final reduction function f :

- Given (G, k) for VC, $G = (V, E)$

- for each edge $(u, v) \in E$

- create vertex w_{uv}

- add new edges $(u, w_{uv}) + (v, w_{uv})$

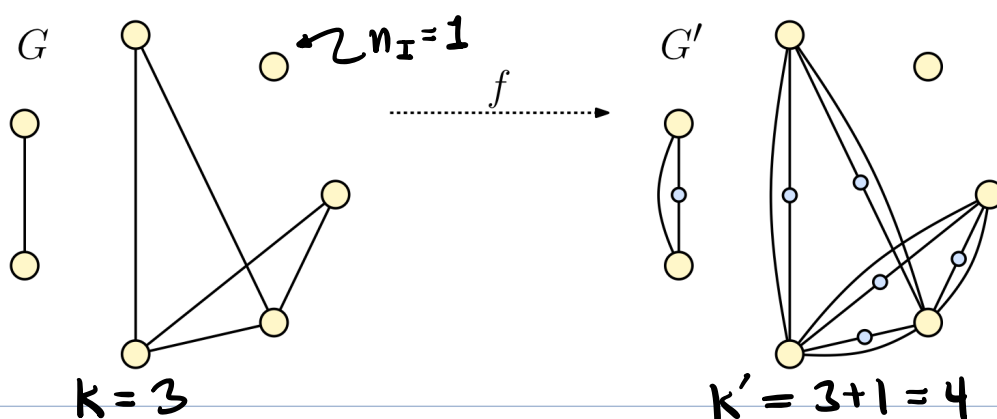


- Call resulting graph G'

- let n_I = no. isolated vertices in G

- set $k' \leftarrow k + n_I$

- return (G', k')



Lemma: G has a vertex cover of size k iff
 G' has a dom. set of size k'

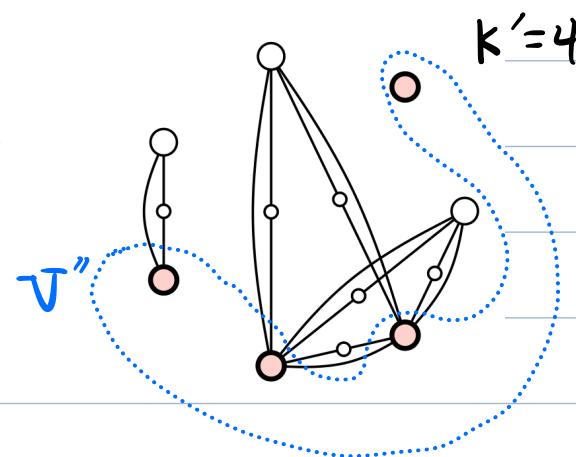
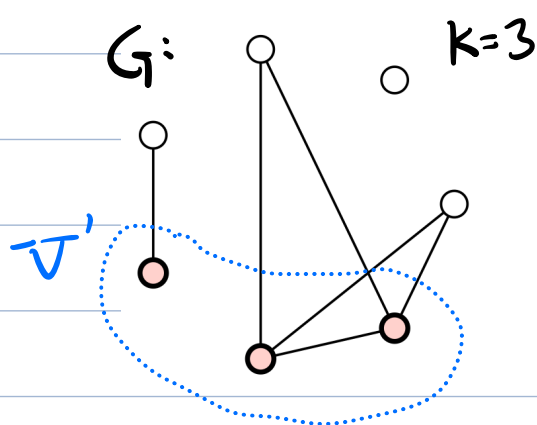
Proof:

$(\Rightarrow)$ Let V' be a VC in G of size k

- Let:

$$V'' = V' \cup \{ \text{all isolated vertices} \}$$

- Clearly, $|V''| = k + n_I = k'$



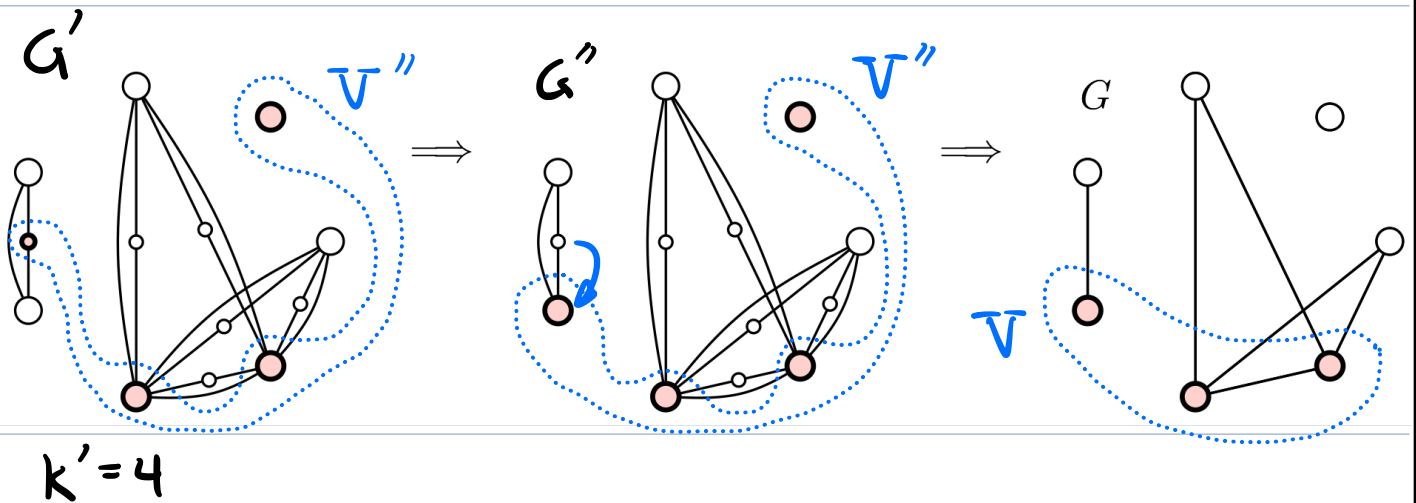
- Since V' is VC, $\forall (u,v) \in E$, either
 $u \in V'$ or $v \in V'$

- In either case we dominate u, v, w_{uv}

$\Rightarrow V''$ is dom. set of size k' in G' $\checkmark$

($\Leftarrow$) Let V'' be a D.S. in G' of size k'

- If V'' contains any mid-edge vertices w_{uv} , replace with either u or v .
- Still a dom.set (as observed earlier)



- Let $V \leftarrow V''$ with isolated vertices removed (Isolated vertices must be in dom.set)

$$|V| = k' - n_I = k$$

- Since V'' dominated all vertices
 - $\Rightarrow V''$ dominates all mid-edge vertices
 - $\Rightarrow V$ covers all edges

$\Rightarrow V$ is a vertex cover in G of size k $\checkmark$

□

Theorem: DS is NP-complete

Proof: (i) $DS \in NP$

Given input $(G=(V,E), k)$

certificate = list of k vertices $\rightarrow V'$

verification:

for each $v \in V$

check that v or some neigh. in V'

if yes for all - accept

else - reject

(ii) $VC \leq_p DS$

Follows from previous lemma.

□

Summary: Three new NP-Complete problems

- CLIQUE
- Vertex cover (VC)
- Dominating Set (DS)