Practice Problems 10

Problem 1. Number of augmenting paths needed

Consider the following flow network $G = (V, E)$ shown in Fig. 1, with source s and sink t. Each edge is labeled with its capacity.

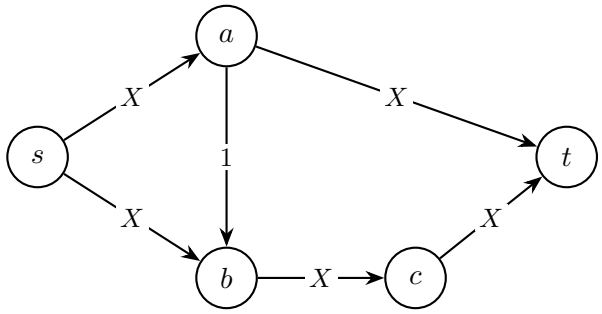


Figure 1: A flow network with source s and sink t.

(a.) Give a sequence of augmenting paths that Ford-Fulkerson could choose on G so that the algorithm uses exactly $2X$ augmentations before terminating. For each path, list its vertices and its bottleneck capacity. Draw the residual graph after the first augmentation and after the second.

(b.) What is the fewest possible augmenting paths the algorithm could possible use on G? Simulate the fattest-path augmenting path algorithm on G, does it reach this bound?

(c.) Change the capacity of edge ab to k. What is the most number of augmentations the Ford-Fulkerson algorithm could use now?

Problem 2. Source Capacity

In class, we showed that if all capacities are integers, then the Ford-Fulkerson algorithm runs for at most C iterations, where C is the sum of all capacities. Let C_s be the sum of all capacities leaving the sink s.

(a.) Explain while C_s is also an upper bound on the number of iterations.

(b.) Draw a graph G for which C_s is much larger than the maximum number of iterations Ford-Fulkerson could ever use.

Problem 3. In lecture, a *circulation with demands* has a demand d_v at each vertex, and a feasible circulation must satisfy $f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$ exactly. In practice, demands are often flexible. Suppose we are given a directed graph $G = (V, E)$ with integer capacities $c_e \geq 0$, and for each vertex v an integer interval $[\ell_v, u_v]$ with $\ell_v \leq u_v$ (the endpoints may be negative). A *feasible flexible circulation* is a function $f : E \to \mathbb{R}_{\geq 0}$ such that

- $0 \leq f(e) \leq c_e$ for every edge e, and
- $\ell_v \leq f^{\text{in}}(v) - f^{\text{out}}(v) \leq u_v$ for every vertex v.

For example, a node that can send out up to 6 units has interval $[-6, 0]$, and a node that needs between 4 and 5 units has interval $[4, 5]$.

(a.) Prove that if a feasible flexible circulation exists, then $\sum_v \ell_v \leq 0 \leq \sum_v u_v$. *Hint:* what is $\sum_v \big(f^{\text{in}}(v) - f^{\text{out}}(v)\big)$ for any f?

(b.) Show how to decide whether a feasible flexible circulation exists by constructing a network G' for the *ordinary* circulation with demands problem. Your G' should add only one new vertex to G, and $O(n)$ new edges. Prove that G has a feasible flexible circulation if and only if G' has a feasible circulation.

Hint: The net inflow at each vertex is no longer fixed, but part (a) tells you something about their sum that *is* fixed.

(c.) Give the running time of your algorithm, assuming you solve the circulation problem on G' with Edmonds-Karp.