Practice Problems 10

Problem 1. Number of augmenting paths needed

Consider the following flow network $G = (V, E)$ shown in Fig. 1, with source s and sink t. Each edge is labeled with its capacity.

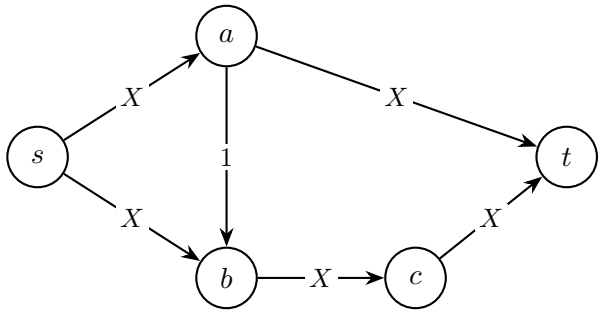


Figure 1: A flow network with source s and sink t.

(a.) The Ford-Fulkerson algorithm could choose the path $s \to a \to b \to c \to t$, then $s \to b \to a \to t$ in an alternating fashion X times, for $2X$ total augmentations. The edge between a and b in the residual would change in direction every time, but it would always limit the capacity of the augmenting path to 1, increasing the s, t flow by 1 on each of the $2X$ iterations.

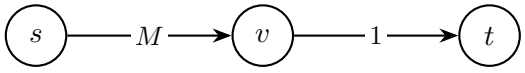
(b.) The algorithm could perform 2 iterations: It could initially choose $s \to a \to t$ with capacity X, and $s \to b \to c \to t$ with capacity X. The fattest-path augmenting path algorithm would choose these paths and saturate the bound.

(c.) If the capacity of ab is k, assuming that $X >> k$, it is not possible for a path to have capacity $< k$. Thus, the flow must increase by k each time, so no more than $\lceil 2X/k \rceil$ augmenting paths may be chosen.

Problem 2. Source Capacity

(a.) The s, t flow is equal to the amount of flow leaving the source s. Because the s, t flow must increase by at least 1 on each augmentation, there may be no more than C_s augmentations.

(b.) Draw a graph G for which C_s is much larger than the maximum number of iterations Ford-Fulkerson could ever use.



In this graph, Ford-Fulkerson could only use 1 iteration, as the maximum s, t-flow is 1. However, C_s is $M >> 1$.

Problem 3. Throughout, for a function f on the edges we write $\text{net}_f(v) = f^{\text{in}}(v) - f^{\text{out}}(v)$ for the net inflow at v, and $n = |V|$, $m = |E|$.

(a.) First we show that $\sum_{v \in V} \text{net}_f(v) = 0$ for any function $f : E \to \mathbb{R}$. Consider a single edge $e = (u, w)$. It contributes $f(e)$ to $f^{\text{in}}(w)$ and $f(e)$ to $f^{\text{out}}(u)$, and appears nowhere else in the sum. So its total contribution to

$$\sum_{v \in V} \text{net}_f(v) = \sum_{v \in V} f^{\text{in}}(v) - \sum_{v \in V} f^{\text{out}}(v)$$

is $f(e) - f(e) = 0$. Summing over all edges, the whole sum is 0.

Now let f be a feasible flexible circulation. Then $\ell_v \le \text{net}_f(v) \le u_v$ for every v. Summing these inequalities over all $v \in V$,

$$\sum_{v \in V} \ell_v \;\le\; \sum_{v \in V} \text{net}_f(v) \;=\; 0 \;\le\; \sum_{v \in V} u_v.$$

(b.) In an ordinary circulation, the net inflow at each vertex is fixed. Here, the net inflow at v may be anywhere in $[\ell_v, u_v]$. We write it as

$$\text{net}_f(v) = \ell_v + g_v, \qquad 0 \le g_v \le u_v - \ell_v,$$

where g_v is the "extra" inflow beyond the minimum. We fix the demand of v at ℓ_v and route the extra g_v units out of v along a new edge to a new vertex z. The amounts g_v are not fixed individually, but by part (a) their sum is:

$$\sum_v g_v = \sum_v \text{net}_f(v) - \sum_v \ell_v = -\sum_v \ell_v.$$

So z can be given a fixed demand.

For the construction:

Let $L = \sum_{v \in V} \ell_v$. Build $G' = (V', E')$ as follows.

- $V' = V \cup \{z\}$, where z is a new vertex.
- E' contains every edge of E, with the same capacity c_e.
- For each $v \in V$, add a new edge (v, z) with capacity $u_v - \ell_v \ge 0$.
- Give each $v \in V$ demand $d'_v = \ell_v$, and give z demand $d'_z = -L$.

This adds one vertex and n edges. The demands satisfy $\sum_{v \in V'} d'_v = L - L = 0$. All capacities and demands are integers.

We claim G has a feasible flexible circulation if and only if G' has a feasible circulation (with demands d').

[Proof ($\Rightarrow$)] Let f be a feasible flexible circulation in G. Define f' on E' by

$$f'(e) = f(e) \text{ for } e \in E, \qquad f'(v, z) = g_v := \text{net}_f(v) - \ell_v \text{ for } v \in V.$$

We check that f' is a feasible circulation in G'.

Capacities. For $e \in E$, $0 \le f'(e) = f(e) \le c_e$. For a new edge, $\ell_v \le \text{net}_f(v) \le u_v$ gives $0 \le g_v \le u_v - \ell_v$.

Demands at $v \in V$. The edges of G' at v are those of G plus the outgoing edge (v, z). So

$$\mathrm{net}_{f'}(v) = \mathrm{net}_f(v) - g_v = \ell_v = d'_v.$$

Demand at z. Every edge at z enters z, so, using part (a),

$$\mathrm{net}_{f'}(z) = \sum_{v \in V} g_v = \sum_{v \in V} \mathrm{net}_f(v) - \sum_{v \in V} \ell_v = 0 - L = d'_z.$$

[$\Leftarrow$] Let f' be a feasible circulation in G', and let f be its restriction to E. For $e \in E$ we have $0 \le f(e) = f'(e) \le c_e$. For $v \in V$, let $g_v = f'(v, z)$, so $0 \le g_v \le u_v - \ell_v$. As above,

$$\mathrm{net}_f(v) = \mathrm{net}_{f'}(v) + g_v = \ell_v + g_v,$$

and therefore $\ell_v \le \mathrm{net}_f(v) \le u_v$. So f is a feasible flexible circulation in G.

(c.) The network G' has $n + 1$ vertices and $m + n$ edges. The max-flow network built from it adds s^*, t^*, and at most one edge per vertex of G', so it has

$$N = n + 3 \text{ vertices} \qquad \text{and} \qquad M \le m + 2n + 1 = O(m + n) \text{ edges.}$$

Building it takes $O(m + n)$ time. Edmonds–Karp runs in $O(NM^2)$ time, so the total running time is

$$O\big(n(m + n)^2\big).$$

If $m = \Omega(n)$ (for example, if the underlying undirected graph of G is connected), this is $O(nm^2)$. In particular, the running time does not depend on the capacities or the intervals, so the algorithm is strongly polynomial.