CMSC 451 Sep 2.

1. Asymptotic Complexity

2. Basic Graph Definitions

3. DFS.

Asymptotic Notation

$$T_1(n) = \boxed{3n^2 + 2n + 10\log n + 5}$$

$$T_2(n) = \max(4n(\log n)^4,\ 6n^3 - 3n\log n)$$

$$T_1(n) = O(n^2) \qquad T_2(n) = O(n^3)$$

Definition (Big-O Notation).

$f(n)$ is $O(g(n))$. if $\exists\, c > 0$, $n_0 \geq 0$

s.t. $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$

$f(n)$ grows slower or equal to $g(n)$ (up to a constant)

1. only talk about n is large
2. ignores constant $\quad 100n^2 = O(n^2)$

   $\quad 10^{10} n^2 = O(n^2)$

$f(n) = O(g(n)) \longrightarrow f(n)$ "$\leq$" $g(n)$

$f(n) = o(g(n))$ $\quad$ $f(n)$ "$<$" $g(n)$

$\forall c > 0, \exists n \geq n_0.\ f(n) \leq c\, g(n)$

$$\lim_{n \to +\infty} \frac{f(n)}{g(n)} = 0$$

$f(n) = \Theta(g(n))$ $\quad$ $f(n)$ "$=$" $g(n)$

$$\lim_{n \to +\infty} \frac{f(n)}{g(n)} = c$$

$f(n) = \Omega(g(n))$ $\quad$ $f(n)$ "$\geq$" $g(n)$

$$\lim_{n \to +\infty} \frac{f(n)}{g(n)} = c \text{ or } +\infty$$

$f(n) = \omega(g(n))$ $\quad$ $f(n)$ "$>$" $g(n)$

$$\lim_{n \to +\infty} \frac{f(n)}{g(n)} = +\infty$$

Find that most important term.

$n^2, n^3, n^{100}, \ldots$ polynomial.

$\log n$. $(\log n)^2 = \log^2 n$, $\log^{100} n$. polylogarithmic.

$2^n, \ldots, 100^n, 1.01^n$. exponential.

exponential $>$ polynomial $>$ polylogarithmic

$$n^3 + \log^4 n + 10 = O(n^3)$$

$$n^3 + \log^4 n + 10 + 1.01^n = O(1.01^n)$$

Graphs.

A graph: structure of objects (vertices, nodes, ...)
& relation between objects. (edges, links, arcs)

$$G = (V, E)$$

set of vertices, set of edges.

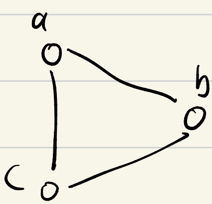
(undirected) graph: an edge is unordered pair of vertices.

$u, v \in V$, $(u, v) \in E$. $(u, v) = (v, u)$

$\{u, v\}$

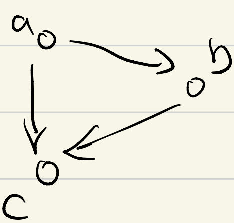
directed graph (digraph): an edge is ordered pair of vertices

$(u, v) \in E$. $(u, v) \neq (v, u)$



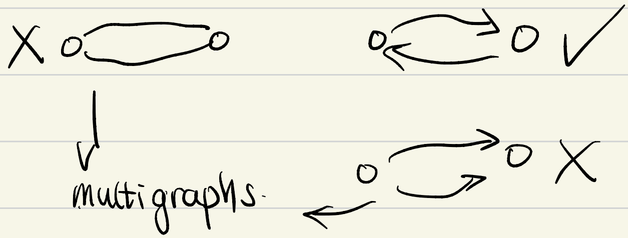
→ undirected graph $V = \{a, b, c\}$

$E = \{(a, b), (b, c), (a, c)\}$

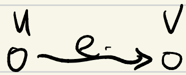


digraph $E = \{(a, b), (b, c), (a, c)\}$

no multiple edges between vertices.

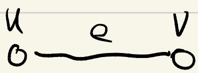


Terminology:



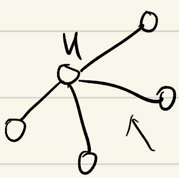
u: is the origin of e

v: destination



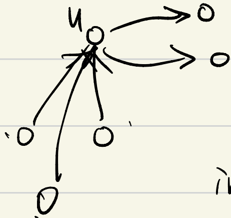
e: is incident on u & v.

u: is adjacent to v.



degree: # edges incident on a vertex.

deg(u) = 4.



in-degree: # edges to the vertex

out-degree: # edges from the vertex

in-deg (u) = 3 , out-deg (u) = 2

$$G: \quad \sum_{u \in V} \deg(u) = 2|E| \qquad u \overset{e}{\text{———}} v$$

$$\sum_{u \in V} \text{in-deg}(u) = |E| \qquad u \xrightarrow{e} v$$

$$\sum_{u \in V} \text{out-deg}(u) = |E|$$

$G = (V, E)$, $\quad |V| = n$. $\quad |E| = \underline{m}$

G undirected. $\quad$ maximal $\underline{m}$ related to n.

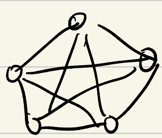
$$0 \leq m \leq \binom{n}{2} = \frac{n(n-1)}{2}$$

G directed $\quad 0 \leq m \leq n^2$

G is <u>sparse</u> if $m = O(n)$

<u>dense</u> if $m = O(n^2)$

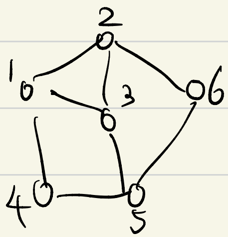
Complete graph: $\quad m = \binom{n}{2}$., $\quad K_n$.

$K_5 =$ 

Path : Sequence of vertices,

$v_1, v_2, \ldots, v_K$, s.t. $(v_i, v_{i+1}) \in E$

length of path = $k-1$.



1 — 2 — 6 — 5 — 4

length = 4.

path from v_1 to v_K.

Cycle : Path s.t. $v_1 = v_K$

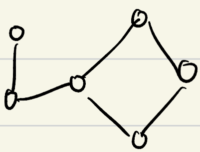
Simple path : no vertex is repeated in the path.

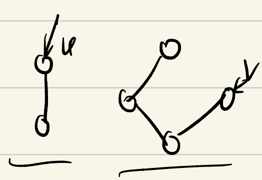
Simple cycles : no vertex is repeated but v_1 & v_K.

$u, v \in V$, u is reachable from v. if

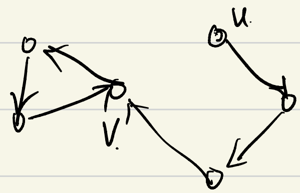
$\exists$ path from v to u.

Graph G is connected if all vertex pairs are reachable.





Digraph is strongly connected: if
$\forall\, u, v \in V.\quad u \rightsquigarrow v \ \&\ v \rightsquigarrow u.$



$u \rightsquigarrow v$

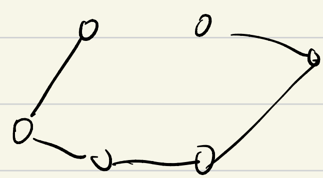
$v \not\rightsquigarrow u$ → not strongly connected.

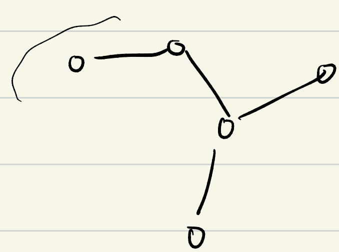
Q:

$G=(V,E),\quad |V|=n.$

What is min $|E|$, s.t. G is connect

A: $n-1$



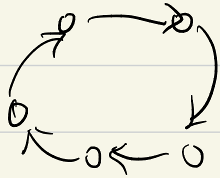


(unrooted)

Tree: $(n-1)$ edges, n vertices connected graph.

Q: What is min $|E|$ in a digraph s.t. G is strongly-connected

A: n.



Representations of Graph.

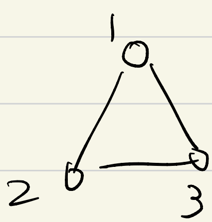
$G = (V, E)$, $n = |V|$ $m = |E|$.

$V = \{1, \dots, n\}$.

Adjacency Matrix of G.

A = $n \times n$ matrix,

$$A[u, v] = \begin{cases} 1, & \text{if } (u,v) \in E \\ 0, & \text{if } (u,v) \notin E \end{cases}$$



A	1	2	3
1	0	1	1
2	1	0	1
3	1	1	0

Pros: 1. easy to see $(u,v) \in E$

2. use it as a matrix.

$A^2[u,v]$ = # paths of length 2 in G, from u to v.

A^k — — — — — — length k.

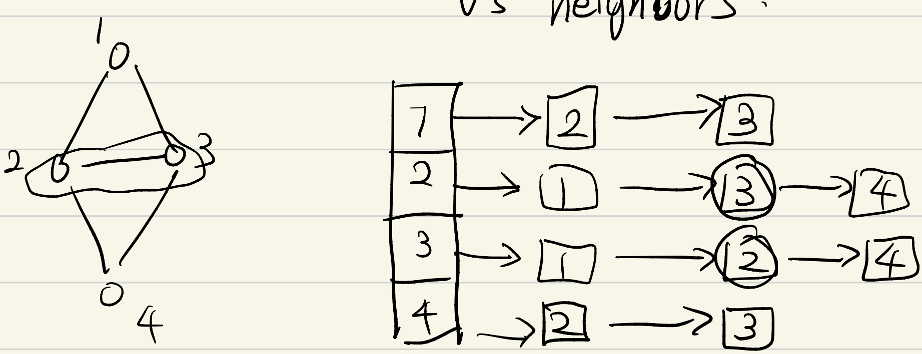
eigen value of A, = spectrum of G.

Cons: $\Omega(n^2)$ space, bad for sparse graphs

Adjacency List:

Array of length n, $A[n]$.

$A[v]$ is head of a linked list of v's neighbors.



Pro: Space: $O(m+n)$

Con: hard to directly find an edge.