Quiz 1 9/21

Cover: 8/31 ~ 9/14 lectures.

Notes, Practice problems, Textbooks.

CMSC 451   9/14

Shortest Path

Weighted ~~graph~~ digraph: $G = (V, E)$ digraph.

$w(u,v)$: weight of $(u,v) \in E$

cost of a path = sum of edge weights.

$(u) \xrightarrow{3} \bigcirc \xrightarrow{5} \circ \xrightarrow{-1} (v)$ cost = 7

distance from u to v, $\delta(u,v)$ =

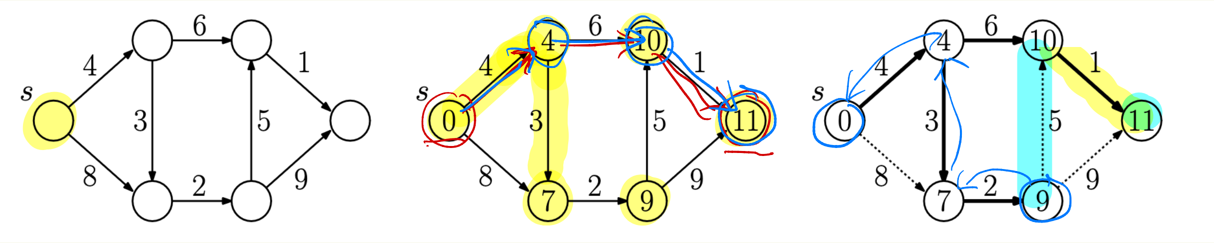
minimum cost of any path from u to v.

single-pair? as hard as single-source (worst case)

Single-source: for a given source $s \in V$,

compute $\delta(s,u)$ for all $u \in V$.

shortest paths form a tree.



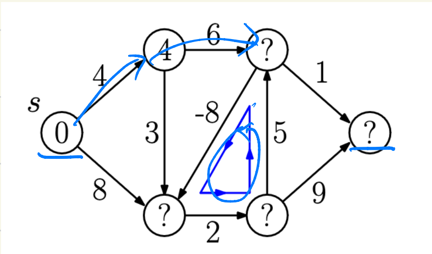
Shortest-path tree.

$pred[u] = v \iff \delta(s,u) = \delta(s,v) + w(v,u)$

Observation: $s \rightsquigarrow u \rightsquigarrow v$:

If $s \rightsquigarrow v$ is a shortest path, then $s \rightsquigarrow u$ is also a shortest path from $s \to u$.

Negative Weights?



negative weight cycle

$\Downarrow$

distance is not well-defined

"Relax" operation

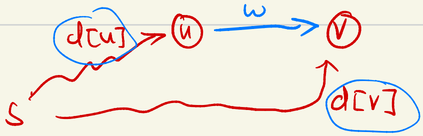
$d[v]$: estimation of $\delta(s,v)$ up to our knowledge.

initially $d[s]=0$, $d[v]=+\infty$ for $v \neq s$.

algorithms update d using edge weights.

via relaxation

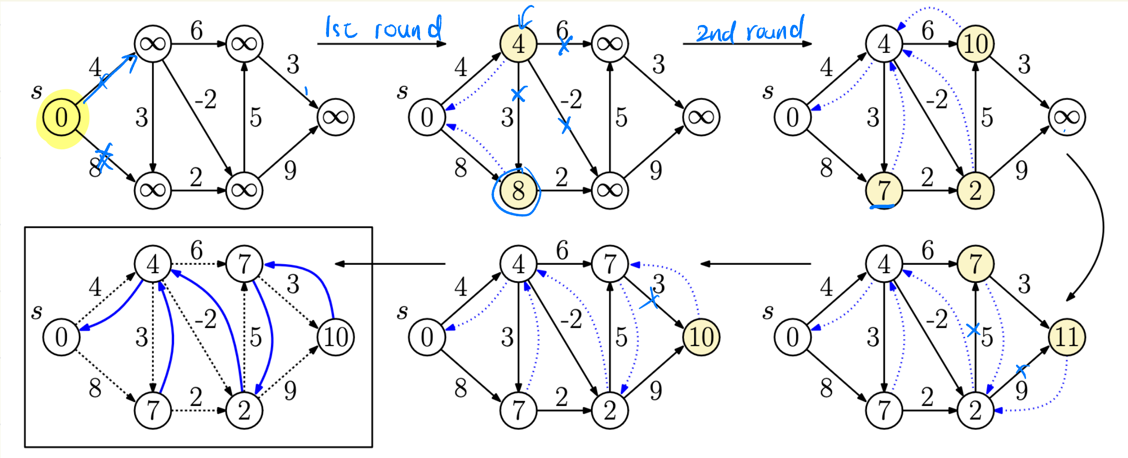
relax (u,v): $(u,v) \in E$	if $d[u] + w(u,v) < d[v]$: $d[v] \leftarrow d[u] + w(u,v)$



$\forall (u,v) \in E,$
$d[u] + w(u,v) \geq d[v]$

Bellman - Ford

Idea : Repeat relaxation until convergence

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Bellman-Ford( G = (V, E), w, s )
    for each v ∈ V,  d[v] ← +∞
    d[s] ← 0
    repeat
        converged ← true
        for each (u,v) ∈ E      ← in any order
                                → O(m)
            relax (u, v)
            if d[v] changed
                converge ← false
    until converged
```

$n = |V|$, $m = |E|$

Running time: $O(nm)$

each relaxation round is $O(m)$

at most $n-1$ rounds (see below)

Correctness:

Idea: each round will propagate shortest paths by one more edge.

$$s \xrightarrow{1} \circ \xrightarrow{2} \circ \xrightarrow{5} \circ \xrightarrow{-1} \circ$$

	0	∞	∞	∞	∞
1st round	0	1	∞	∞	∞
2nd round	0	1	3	∞	∞
3rd	0	1	3	8	∞
4th	0	1	3	8	7

Lemma: If G has no negative cycles, Bellman-Ford will terminate within $n-1$ rounds with $d[v] = \delta(s, v)$

at most $(n-1)$ edges in shortest path.

Claim: After k-th round, for all $u \in V$ s.t. shortest path $s \to u$ has at most k edges,
$d[u] = \delta(s, u)$

Proof: by induction over k.

Dijkstra Algorithm

Greedy algorithm

Requires edge weights ≥ 0

Idea: relax from closest vertex

Dijkstra $(G=(V,E), w, s)$

for each $v \in V$, $d[v] \leftarrow +\infty$

$d[s] \leftarrow 0$

(of all vertices)

$Q \leftarrow$ priority queue sorted by d-values,

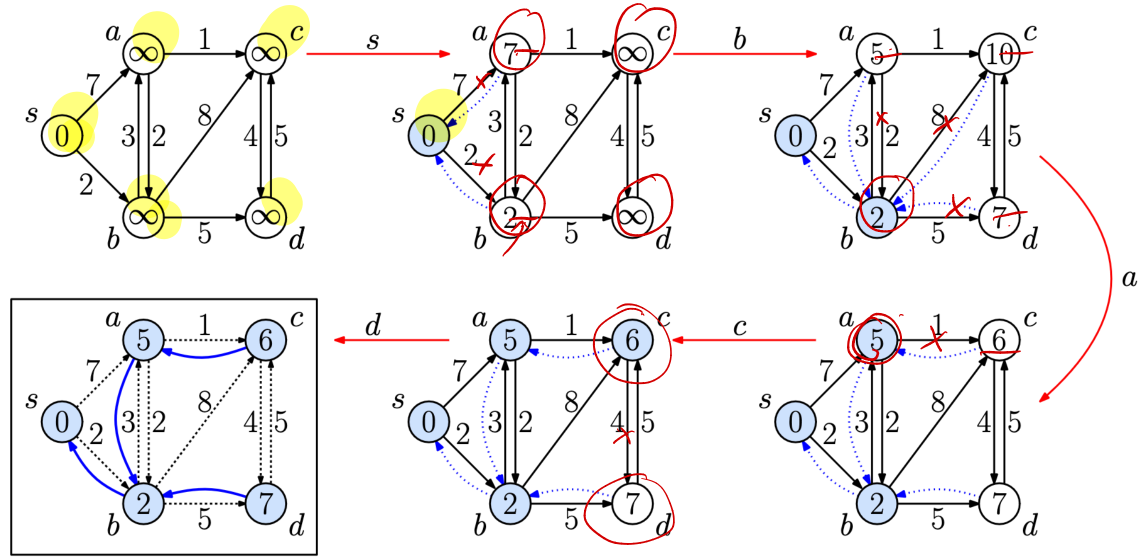
while Q is not empty: ← $O(n)$ rounds

  $u \leftarrow Q$.extractmin( )

  for each $(u,v) \in E$ ← out-deg(u) times

    relax(u,v)

    update Q if d[v] changes



Running time:

extract min() : $O(n)$ times

relax() : $\sum_{v}$ out-deg(v) $= O(m)$ times
(update())

$O((n+m)\log n)$ or $O(n\log n + m)$

Correctness:

When u get picked from Q, $d[u] = \delta(s, u)$

Proof. Assume u get picked,

we have $d[u] \geq \delta(s, u)$

$$\left(\underbrace{s \rightsquigarrow x}_{\text{picked}} \right) \to \underbrace{y \rightsquigarrow u}_{\text{not picked}} \quad \text{shortest path}$$

$d[x] = \delta(s, x)$ (by induction)

when relaxing (x, y)

$d[y] = \delta(s, x) + w(x, y) = \delta(s, y)$

$\therefore\ d[u] \underset{\uparrow}{\leq} d[y] = \delta(s, y) \leq \delta(s, u)$

u is min-d among unselected

$\therefore\ d[u] = \delta(s, u)$