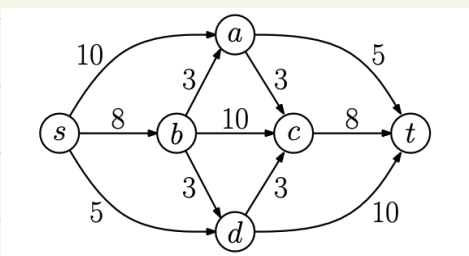
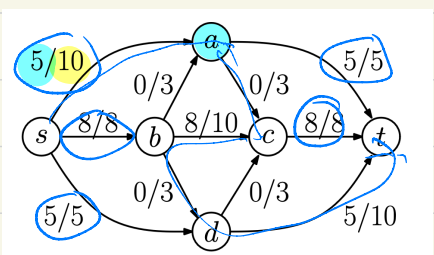
CMSC 451 Oct. 26

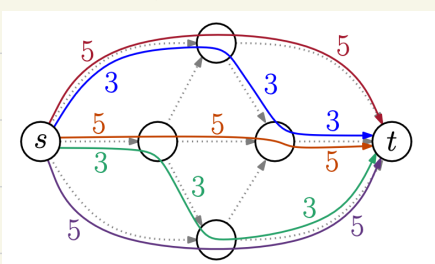
Network Flow (II)

- Flow networks: capacity, source, sink (terminal)
- Flows: capacity constraint
  flow conservation

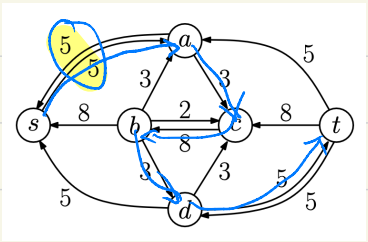




- Path view vs. edge view



- Residual Network & Augment Paths.



(u) $f(u,v)/c(u,v)$ (v)

↓ in G_f

(u) $c(u,v)-f(u,v)$ / $f(u,v)$ (v)

- Ford-Fulkerson Algorithm

Today: Max-Flow Min-Cut Theorem

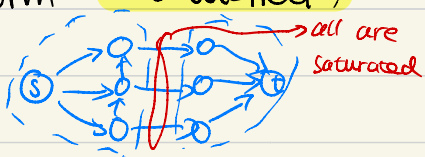
Recall: In Residual Network G_f.
if ∃ (non-zero) path $s \to t$ (Augment Path).
then f can be improved into $f + f'$

if ~~∃~~ path $s \to t$ / no augment path
do we find the max flow?

Claim: Given flow network G and flow f.
If no path $s \to t$, in G_f,
then f is max flow.

Proof idea: using the concept of "cut".

Intuition: a max flow must has some saturated edges, that form a bottleneck, splitting s from t.

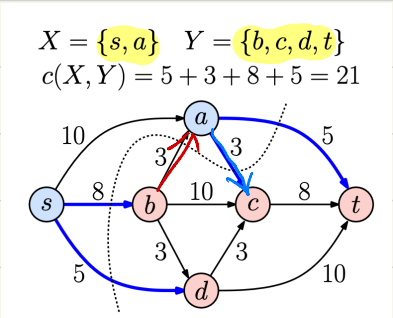


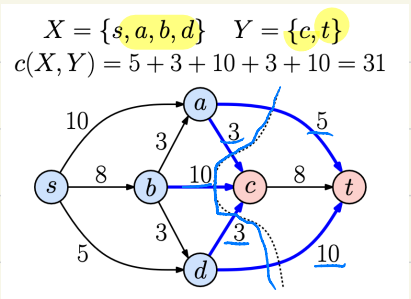
Definition: Given a flow network G, a cut is a partition of vertices into X & Y, s.t. $s \in X$ & $t \in Y$.

The capacity of cut (X, Y) is the sum of capacities from X to Y.

$$c(X, Y) = \sum_{\substack{(u,v) \in E \\ u \in X \\ v \in Y}} c(u, v)$$

max amount of flow from X to Y.



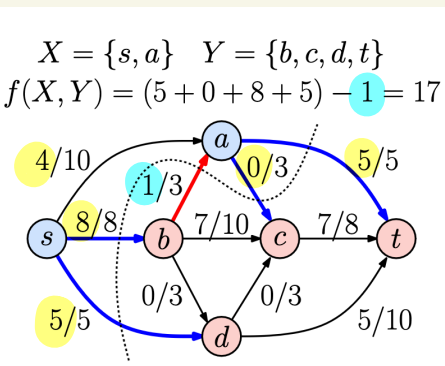


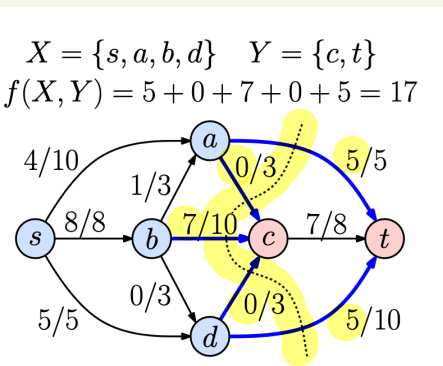
Define flow through the cut (X, Y) to be

amount of flow $X \to Y$ $-$

amount of flow $Y \to X$

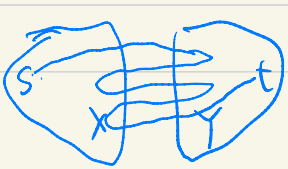
$$\sum_{\substack{(u,v)\in E \\ u\in X \\ v\in Y}} f(u,v) - \sum_{\substack{(u,v)\in E \\ u\in Y \\ v\in X}} f(u,v)$$





Theomem, $\forall$ cut (X, Y),

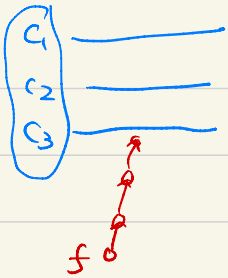
$$f(X, Y) = |f|$$



proof using flow conservation.

Claim: $\forall$ cut (X, Y), flow f,

$$|f| = f(X, Y) \leq c(X, Y)$$

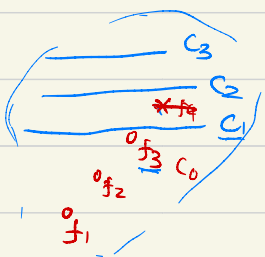


$$\max_{f:\text{flow}} |f| \leq \min_{(X,Y)\text{ cut}} c(X, Y) \quad : (\text{max flow} \leq \text{min cut})$$

$\Rightarrow$ if some flow f and some cut (X, Y) s.t. $|f| = c(X, Y)$

then f must be max flow

(X, Y) must be min cut



Claim: Given flow network G and flow f,
If no path $s \to t$ in G_f,
then f is max flow.

Todo: find a cut (X, Y) s.t
$c(X, Y) = |f|$.

Claim: Given flow network G and flow f.
If no path $s \to t$, in G_f,
then f is max flow.

Proof: No path $s \to t$ in G_f,

$\Rightarrow$ Find a cut of G.

Let X to be all vertices that
s can reach in G_f.
Y to be other vertices.

Since $s \not\to t$, (X, Y) is a cut.

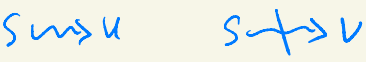
already proven

Claim: $|f| = f(X,Y) = c(X,Y)$

$$f(X,Y) = \sum_{\substack{(u,v)\in E \\ u\in X \\ v\in Y}} f(u,v) - \sum_{\substack{(u,v)\in E \\ u\in Y \\ v\in X}} f(u,v)$$

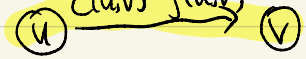
11?

$$c(X,Y) = \sum_{\substack{(u,v)\in E \\ u\in X \\ v\in Y}} c(u,v)$$



Case #1. $(u,v) \in E$, $u \in X$, $v \in Y$,

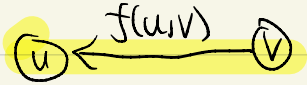
$f(u,v)$ must equal $c(u,v)$.

otherwise there is 

in G_f $\Rightarrow$ s can reach v in G_f, contradiction.

Case #2. $(u,v) \in E$, $u \in Y$, $v \in X$.

$f(u,v)$ must be 0.

otherwise there is 

in G_f $\Rightarrow$ s can reach u in G_f, contradiction

Thus,

$$|f| = f(X,Y) = \sum_{\substack{(u,v)\in E \\ u\in X \\ v\in Y}} f(u,v) - \sum_{\substack{(u,v)\in E \\ u\in Y \\ v\in X}} f(u,v)$$

$$= \sum_{\substack{(u,v)\in E \\ u\in X \\ v\in Y}} c(u,v) - 0$$

$$= c(X,Y)$$

$\Rightarrow$ f is max flow.

Algorithm for Min-Cut.

1. Run Ford-Fulkerson, get a max flow. f

2. $X \Leftarrow$ vertices that s can reach in G_f

   $Y \Leftarrow$ " " " can't.

3. (X, Y) is the min cut.