CMSC 754: Fall 2026 Dave Mount

Homework 3

Due by the start of class on Tuesday, Oct 6. (Submissions will be through Gradescope.) Late homeworks are not accepted (unless an extension has been prearranged) so please turn in whatever you have completed by the due date. Unless otherwise specified, you may assume that all inputs are given in *general position*.

Problem 1. Explain how to solve each of the following problems in linear (expected) time. Each can be modeled as a linear programming (LP) problem, perhaps with some additional pre- and/or post-processing.

In each case, explain how the problem is converted into an LP instance and how the answer to the LP instance is used/interpreted to solve the stated problem. (See the note at the end of this handout on guidance in presenting LP reductions.)

(a) You are given two point sets $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_m\}$ in the plane, and you are told that they are separated by a vertical line $x = x_0$, with P to the left and Q to the right (see Fig. 1(a)). Compute the line equations of the two "crossing tangents," that is, the lines ℓ_1 and ℓ_2 that are both supporting lines for $\text{conv}(P)$ and $\text{conv}(Q)$ such that P lies below ℓ_1 and above ℓ_2 and the reverse holds for Q. (Note that you are *not* given the hulls, just the point sets.) Your algorithm should run in time $O(n + m)$.

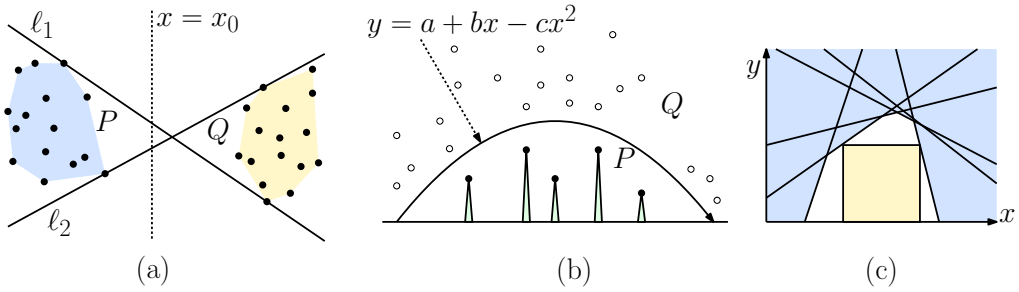


Figure 1: LP applications.

(b) You have a cannon in $\mathbb{R}^2$. It has three controls labeled "a," "b," and "c". A projectile shot from this cannon travels along the parabolic arc $y = a + bx - cx^2$. You are asked to determine whether it is possible to adjust the controls so that the projectile travels above a set of n building tops, represented by a point set $P = \{p_1, \ldots, p_n\}$ and beneath a set of m floating balloons, represented by a point set $Q = \{q_1, \ldots, q_m\}$ (see Fig. 1(b)). Your algorithm should run in time $O(n+m)$. (I do not care where the cannon is actually located. If your solution is based on some assumption about the cannon's location, please state this.)

(c) You are given a set of n halfplanes $H = \{h_1, \ldots, h_n\}$, where h_i is given as a pair (a_i, b_i) and it consists of all the points of the plane that lie on or beneath the line $y = a_i x + b_i$. Compute the axis-parallel square of the largest side length whose lower edge lies on the x-axis (see Fig. 1(c)). If no such square exists, your algorithm should indicate this.

Problem 2. In this question, we will explore an alternate way in which to define point-line duality in the plane. (This generalizes to any dimension, but the plane is the simplest case to work with.) Any line ℓ in the x, y-plane that does not pass through the origin can be expressed by the equation $\ell : ax + by = 1$. Let us define the *polar dual* of ℓ, denoted ℓ° to be the point (a, b). Similarly, given a point $p = (a, b)$, define its dual to be the line $p^\circ : ax + by = 1$. Clearly, this mapping is an involution, that is, $p^{\circ\circ} = p$ and $\ell^{\circ\circ} = \ell$.

(a) Given a line $\ell : ax + by = 1$, define its *(open) inner halfplane*, denoted $\ell_<$, to be set of points that lie on the same side of ℓ as the origin, that is $(x, y) \in \ell_<$ if $ax + by < 1$ (see Fig. 2(a)). Define $\ell_>$ analogously for points on the opposite side of ℓ, and define $\ell_\leq$ and $\ell_\geq$ analogously for the closed halfplanes. We say that point p is *inside* (resp., *outside*) ℓ if $p \in \ell_<$ (resp. $\ell_>$).

Prove that polar duality is "order reversing" by showing that point p lies inside/on/outside of ℓ if and only if ℓ° lies inside/on/outside of p°, respectively (see Fig. 2(b)).

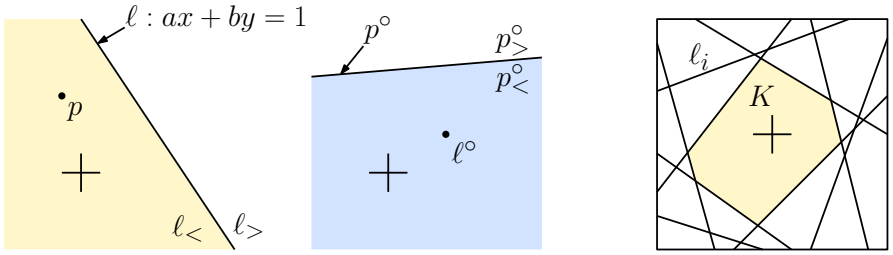


Figure 2: The polar dual.

(b) Prove that polar duality is "incidence preserving" by showing that ℓ_1 and ℓ_2 intersect at point p if and only if the dual line p° passes through dual points ℓ_1° and ℓ_2°.

(c) Let $L = \{\ell_1, \ldots, \ell_n\}$ be a set of lines in $\mathbb{R}^2$ and let K be the intersection of the associated (closed) inner halfplanes, that is, $K = \bigcap_{i=1}^n \ell_\leq$ (see Fig. 2(c). Let's assume that K is not empty (which implies that it contains the origin).

Describe the relationship between K and the convex hull of the polar dual set of points $L^\circ = \{\ell_1^\circ, \ldots, \ell_n^\circ\}$? In particular, explain how the CCW order of edges around the boundary of K relates to the order of points around $\text{conv}(L^\circ)$. Justify your answer.

Problem 3. Euler's formula is useful for computing the combinatorial properties of planar subdivisions. A planar graph (or more accurately, a cell complex) is a subdivision of the plane into vertices (0-dimensional), edges (1-dimensional), and faces (2-dimensional). Let v, e, and f denote the number of vertices, edges, and faces, respectively, in a given cell complex. (Note that f includes the unbounded face that extends to infinity.) Euler's formula states that these quantities are related as

$$2 \;=\; v - e + f$$

For example, in Fig. 3(a) we show a triangulation of a set of $v = 16$ vertices with $h = 10$ vertices on the convex hull. In Fig. 3(b) we show a quadrilateral cell complex. Using Euler's formula, answer each of the following questions.

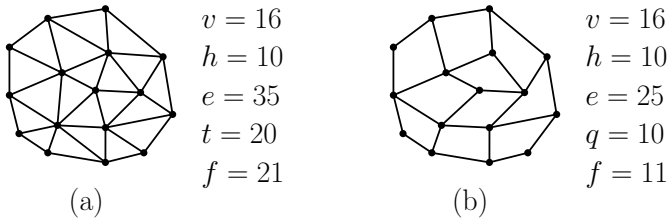


Figure 3: Applications of Euler's formula.

(a) Given a triangulation with v vertices, where h of these lie on the convex hull, use Euler's formula to derive the formula for the number of triangles t and the number of edges e in the triangulation as functions of v and h. (**Hint:** Using the fact that each triangle is incident to three edges, derive a formula that expresses t as an exact function of e and h.)

(b) Given a quadrangulation (a cell complex where each face has four edges, excluding the face lying outside the convex hull) with v vertices, where h of these lie on the convex hull, use Euler's formula to derive the formula for the number of quadrangles q and the number of edges e in the quadrangulation as functions of v and h.

(c) Explain why your answer to (b) implies that a quadrilateralization does not exist if the number of hull vertices is odd.

Problem 4. (Optional - Ungraded) Explain how to solve each of the following problems in linear (expected) time by reducing it to a linear-programming (LP) problem. This may involve multiple instances and/or additional pre- or post-processing.

(a) You are given a set of vertical line segments all in the positive quadrant of the x, y-plane, where the ith segment is specified by giving its center point $c_i = (c_{i,x}, c_{i,y})$ and its height h_i (see Fig. 4(a)).

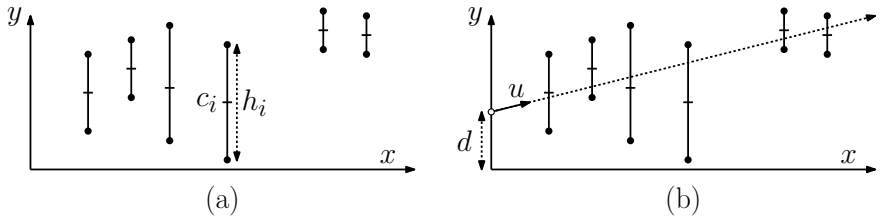


Figure 4: Hitting all segments with a single ray.

Present an algorithm that determines whether there exists a ray that originates on the y-axis and intersects all the segments. If such a ray exists, your algorithm should output the y-coordinate d where the ray originates and the ray's directional vector $u = (u_x, u_y)$ (see Fig. 4(b)). Explain briefly.

(b) Consider the same problem as (a) but with the following modification. If there exists a shot that hits all the segments, output the shot that comes as close as possible (on average) to hitting the centers of the segments. More formally, given any shot, let p_i

denote the point where the shot crosses the ith segment (see Fig. 5). The objective is to minimize the *absolute value* of $\frac{1}{n}\sum_{i=1}^{n}(p_{i,y} - c_{i,y})$. (Note that this is the absolute value of the average, not the average of the absolute values.) Your algorithm should run in $O(n)$ time. (**Hint:** You may simply explain how to modify your solution from (a).)

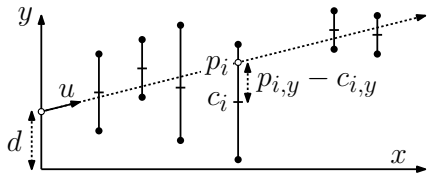


Figure 5: Hitting all segments close to the centers.

Note: Challenge problems are not graded as part of the homework. The grades are recorded separately. After final grades have been computed, I may "bump-up" a grade that is slightly below a cutoff threshold based on these extra points. (But there is no formal rule for this.)

Challenge Problem:

(a) Consider a set Q of n points uniformly distributed in the unit square $[0,1] \times [0,1]$. Prove that the expected number of points in the Pareto set of Q is $\ln n + O(1)$ (see Fig. 6(a)). **Hint**: Sort the points in *decreasing* order of x-coordinates as $\langle q_1, \ldots, q_n \rangle$, where $q_i = (x_i, y_i)$. Prove that the probability that q_i is in the Pareto set is $1/i$. The only property of uniformly distributed points needed is that the x- and y-coordinates of any point are statistically independent of each other.

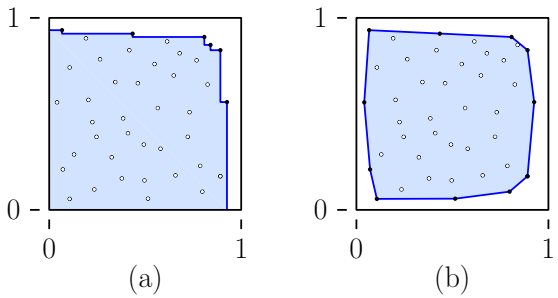


Figure 6: Expected number of points in the (a) Pareto set and (b) convex hull.

(b) Given (a), show that the expected number of points on the convex hull of a set of n uniformly distributed points in the unit square in the plane is $O(\log n)$ (see Fig. 6(b)).

Guidance for Writing LP Reductions: In a linear programming (LP) reduction, you should explain the following:

- how the solution space is modeled as a vector (and in what dimension),
- what are the constraints,

- what is the objective function (and express it as a vector)
- how to interpret the result (including the cases feasible, unbounded, infeasible)

Here is an example.

Sample Problem: Given two sets of points $R = \{r_1, \ldots, r_n\}$ and $B = \{b_1, \ldots, b_n\}$, both in $\mathbb{R}^3$, present an algorithm that determines whether their exists a plane h in $\mathbb{R}^3$ such that all the points of R lie on or above h and all the points of B lie on or below h.

Sample solution: We reduce the problem to linear programming in $\mathbb{R}^3$. Let's assume that each $r_i \in R$ is given in coordinate form as $(r_{i,x}, r_{i,y}, r_{i,z})$ and similarly for B. Let's model h by the equation $z = ax + dy + e$, for some real parameters a, d, and e. To enforce the condition that each r_i lies on or above h and each b_j lies on or below it, we add the constraints

$$\begin{aligned} r_{i,z} &\geq ar_{i,x} + dr_{i,y} + e, \qquad \text{for } 1 \leq i \leq n \\ b_{j,z} &\leq ab_{j,x} + db_{j,y} + e, \qquad \text{for } 1 \leq j \leq n. \end{aligned}$$

We then invoke LP with $2n$ constraints in $\mathbb{R}^3$ (with the variables (a, d, e)). Since this is a yes-no answer, we don't really care about the objective function. We can set it arbitrarily, for example, "maximize e" (which is equivalent to using the objective vector $c = (0, 0, 1)$).

We interpret the LP's result as follows. If the result is "infeasible", then we know that no such plane exists. If the answer is "feasible" or "unbounded", then we assert that such a plane exists (assuming general position). This is clearly true if the result is "feasible", since we can just take h to be the plane associated with the optimum vertex (a, d, e). If the result is "unbounded", then the plane is vertical, but there exists a perturbation such that R lies above and B lies below.

By the way, if you wish to be even more formal (which I rarely require), you can express the LP in standard form as maximizing c^T form as $Ax \leq b$, where x is the symbolic vector $(a, d, e) \in \mathbb{R}^3$, and A and b can be expressed as

$$\begin{bmatrix} r_{1,x} & r_{1,y} & 1 \\ \vdots & \vdots & \vdots \\ r_{m,x} & r_{m,y} & 1 \\ -b_{1,x} & -b_{1,y} & -1 \\ \vdots & \vdots & \vdots \\ -b_{n,x} & -b_{1,y} & -1 \end{bmatrix} \begin{bmatrix} a \\ d \\ e \end{bmatrix} \leq \begin{bmatrix} r_{1,z} \\ \vdots \\ r_{m,z} \\ -b_{1,z} \\ \vdots \\ -b_{n,z} \end{bmatrix}$$