Homework 5

Due by the start of class on Tuesday, Nov 3. (Submissions will be through Gradescope.) Late homework submissions are not accepted (unless an extension has been prearranged), so please turn in whatever you have completed by the due date. Unless otherwise specified, you may assume that all inputs are given in *general position*.

Problem 1. You are given a set $P = \{p_1, \ldots, p_n\}$ of n points in $\mathbb{R}^2$ in general position. For $1 \leq i \leq n-1$, define the ith *critical radius*, denoted r_i, to be the smallest value such that, if we place a Euclidean ball of radius r_i centered at each point of P, the resulting collection of balls forms a set with $n-i$ connected components. Intuitively, the balls grow at the same rate, and we want to find the events where two disconnected groups of balls collide with each other (see Fig. 1).

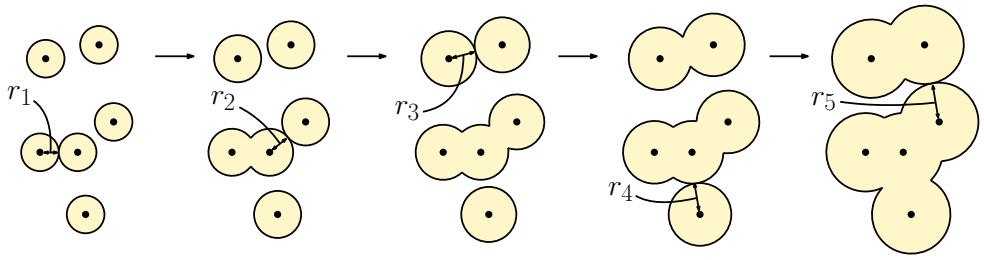


Figure 1: Critical radii.

Present an efficient algorithm which, given P, computes the entire sequence $\langle r_1, \ldots, r_{n-1} \rangle$. Justify your algorithm's correctness and derive its running time. (**Hint:** Aim for a running time of $O(n \log n)$. Be sure that your justification explains clearly why the selected radius values are correct.)

Problem 2. Let P be a set of n point sites in general position in the plane. Consider the following graphs defined on the vertex set P.

Relative Neighborhood Graph: In this graph, denoted $\mathrm{RNG}(P)$, two sites $p, q \in P$ are adjacent if and only if there is no other site of P that is simultaneously closer to p and q than they are to each other. That is, there exists no $r \in P$ such that $\max(\|r-p\|, \|r-q\|) < \|p-q\|$.

Gabriel Graph: In this graph, denoted $\mathrm{Gabriel}(P)$, two sites $p, q \in P$ are adjacent if and only if there is no other site of that lies within the circle whose diameter is $\overline{pq}$.

The objective of this problem is to prove the following string of inclusions:

$$\mathrm{EMST}(P) \subseteq \mathrm{RNG}(P) \subseteq \mathrm{GG}(P) \subseteq \mathrm{DT}(P).$$

To do this, prove each of the following inclusions. (As always, you may assume that points are in general position.)

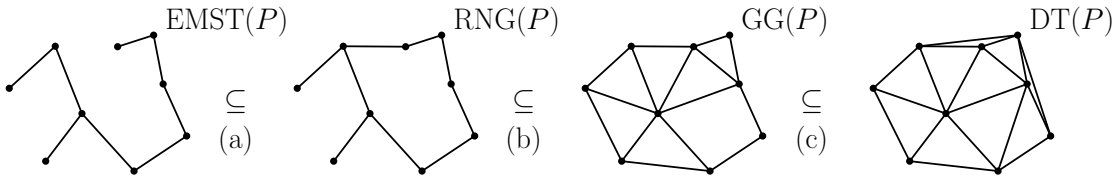


Figure 2: Lunes and the relative neighborhood graph.

(a) Prove that EMST(P) $\subseteq$ RNG(P). (**Hint:** The structure of this proof is structurally similar to the proof given in class that EMST(P) $\subseteq$ DT(P).)

(b) Prove that RNG(P) $\subseteq$ GG(P).

(c) Prove that GG(P) $\subseteq$ DT(P).

Problem 3. This problem presents an alternative proof of the Zone Theorem for line arrangements in the plane. The proof is based on a *token counting game*. Let L be a set of n lines in the plane, and let $\mathcal{A}(L)$ denote its arrangement. Let ℓ be any other line, and recall that its *zone*, denoted $Z_{\mathcal{A}}(\ell)$, is the set of faces that ℓ intersects in $\mathcal{A}(L)$. Recall that the complexity of the zone is the sum of the lengths of all edges of the arrangement that intersect ℓ and bound faces in the zone. (We do not consider ℓ to be part of the arrangement, and so the fact that it appears to split some of these edges is not counted. Note however that an edge of the arrangement that intersects ℓ should be counted twice, once for the left face and once for the right face.)

For illustration, assume ℓ is horizontal. Consider for now just the left-bounding edges of $Z_{\mathcal{A}}(\ell)$ that lie above ℓ. (Recall that an edge is *left bounding* if it lies locally on the left side of a face of the zone.) Place a single token on each such edge. The rules of the game are illustrated in the figure below.

Rule 1: If this edge intersects ℓ, move the token to the point at which the edge intersects ℓ.

Rule 2: If the edge does not intersect ℓ, then let ℓ' denote the line of L supporting this edge, and let ℓ'' be the line whose intersection with ℓ' forms the lower vertex of this edge. Move the token to the intersection of ℓ'' with ℓ.

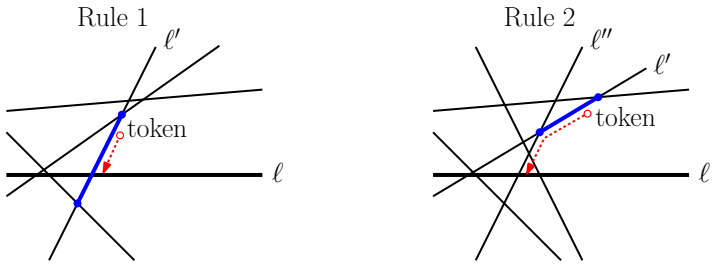


Figure 3: Bead moving rules.

(a) Show that after applying Rules 1 and 2 (just to the left-bounding edges lying in the portion of the zone above ℓ) the total number of tokens that can reside at the intersection of ℓ with any other line of the arrangement is at most 2.

(b) Using (a), show that the total number of edges, summed over all the faces of the zone ℓ (that is, the sum of degrees of the faces of the zone), is at most $8n$.

(c) Explain how to tighten the argument to obtain an upper bound of $6n$ on the total number of edges, summed over all the faces of the zone. (Hint: Some edges are counted multiple times in (b).)

(d) Give an example showing that, for arbitrarily large n, the number of edges on the portion of the zone above ℓ can be as high as $4n - O(1)$. (As a challenge, try to get $6n - O(1)$ total edges in the arrangement, counting the edges of the zone both above and below ℓ. I must admit that I do not see how to do it, although I can get $5n - O(1)$.)

Note that in parts (b)–(d), if an edge is shared between two faces of the zone, it should be counted twice.

Problem 4. (Optional–Ungraded) We start with two (seemingly unrelated) definitions. A triangulation of a set of points in the plane is *acute* if the angles of all its triangles are strictly acute, that is, less than $\pi/2$. (See Fig. 4(a) for an example of a triangulation that is *not* acute.) The Voronoi diagram of a set of points in the plane is said to be *medial* if each edge of the diagram contains in its interior the midpoint of the two defining sites for the edge. (See the Fig. 4(b) for an example of a Voronoi diagram that is *not* medial.)

(a) Prove that any acute triangulation of a set of points in the plane is a Delaunay triangulation, that is, it satisfies the empty circumcircle property.

(b) Prove that if the Voronoi diagram of a set of points in the plane is medial, then the corresponding Delaunay triangulation is acute.

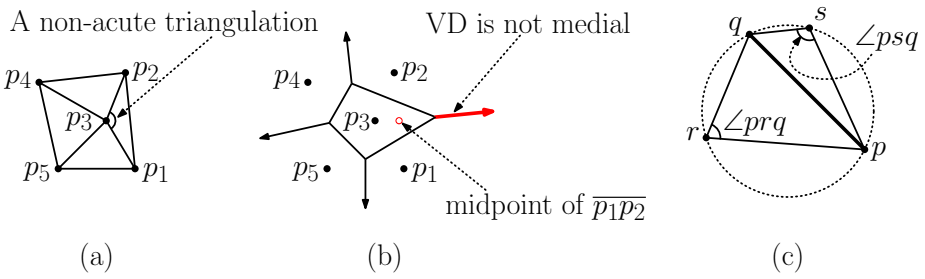


Figure 4: Acute triangulations and medial Voronoi diagrams.

Hint: The following basic geometric fact may be useful. Let $\overline{pq}$ be a chord of a circle and let r and s be two points lying on the circle, one on either side of $\overline{pq}$. Then $\angle prq + \angle psq = \pi$. (See Fig. 4(c).)

Note: Challenge problems are not graded as part of the homework. The grades are recorded separately. After final grades have been computed, I may "bump up" a grade that is slightly below a cutoff threshold based on these extra points. (But there is no formal rule for this.)

Challenge Problem. Let $P = \{p_1, \ldots, p_n\}$ denote a set of n points in $\mathbb{R}^2$, and let $\mathcal{A}(P^*)$ denote the line arrangement defined by the associated dual lines (assuming our standard duality map $(a, b) \to y = ax - b$).

(a) Assume that the origin lies in the interior of some face f_0 of $\mathcal{A}(P^*)$. Describe the subset of points of P that contribute an edge to f_0, that is, $\{p_i \in P \mid p_i^* \cap \partial f_0 \neq \emptyset\}$. As part of your answer, explain how the ordering of edges around the face corresponds to the ordering of the associated points of P. (**Hint:** It may simplify matters to assume that f_0 is bounded, that is, it does not extend to infinity. The set consists of two convex chains. Describe them.)

(b) Generalizing this, consider any point $q \in \mathbb{R}^2$ that lies in the interior of some face f_q of $\mathcal{A}(P^*)$. Describe the subset of points of P that contribute an edge to f_q.