Solutions to Homework 3

Solution 1:

(a) The objective is to compute the two crossing tangents ℓ_1 and ℓ_2 for the convex hulls of two point sets $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_m\}$ that are separated by a vertical line $x = x_0$, with P to the left and Q to the right. Let's assume that $p_i = (p_{i,x}, p_{i,y})$ and $q_j = (q_{j,x}, q_{j,y})$.

We will show that we can compute each of the lines through two applications of LP in RE^2. (It would also be possible to do this with a single application of LP in $\mathbb{R}^4$.) Let's consider ℓ_1 first. The line can be modeled by the equation $y = ax + b$. The points of P lie below ℓ_1 and the points of Q lie above, thus for $1 \le i \le n$ and $1 \le j \le m$ we have:

$$a p_{i,x} + b \;\ge\; p_{i,y} \qquad \text{and} \qquad a q_{j,x} + b \;\le\; q_{j,y}.$$

These inequalities define a collection $n + m$ linear constraints in the two-dimensional space (a, b). The desired line is the one with the largest slope, and so the objective is to maximize the value of a. That is, the objective function is given by the objective vector $(1, 0)$ in (a, b)-space. This is just an instance of linear programming in $\mathbb{R}^2$. We can apply the LP algorithm from class (with $n + m$ constraints in $d = 2$ dimensional space). We assert that, given the problem description, there exists a feasible solution to the LP. It cannot be infeasible under our assumption that P lies to the left of the separating line and Q lies to the right. It cannot be unbounded, assuming that each set has at least one point. The solution for ℓ_2 is symmetrical. Thus, the problem can be solved in expected time $O(n + m)$.

If we want to be more formal (which is not required), we can express the inequalities as maximizing $f^{\mathsf{T}}x$ subject to $Ax \le g$, where $f = (1, 0)^{\mathsf{T}}$ and $Ax \le g$ can be expressed as

$$\begin{bmatrix} -p_{1,x} & -1 \\ \vdots & \vdots \\ -p_{n,x} & -1 \\ q_{1,x} & 1 \\ \vdots & \vdots \\ q_{m,x} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \;\le\; \begin{bmatrix} -p_{1,y} \\ \vdots \\ -p_{n,y} \\ q_{1,y} \\ \vdots \\ q_{m,y} \end{bmatrix}.$$

(b) We want to determine whether there is some assignment of cannon aiming parameters (a, b, c) that satisfies all the constraints. We will ignore the messy issue of where the cannon is located, in order to focus on the principal issue of whether the points of P lie below the parabolic curve and the points of Q lie above it. The conditions that a point $p_i \in P$ lies below the parabola and that the a point $q_j \in Q$ lies above it are

$$a + b p_{i,x} - c p_{i,x}^2 \;\ge\; p_{i,y} \qquad \text{and} \qquad a + b q_{j,x} - c q_{j,x}^2 \;\le\; q_{j,y},$$

for $1 \le i \le n$ and $1 \le j \le m$. (Note that this allows for solutions where the projectile passes through obstacle endpoints. This is an inherent limitation of the fact that the LP requires

that the feasible polytope is a closed set.) From a purely practical standpoint, we could also add the constraint $c \geq 0$, which enforces the condition that gravity bends the cannon ball down, but this is not required.

Doing this for $1 \leq i \leq n$ and $1 \leq j \leq m$ yields $n + m$ linear constraints in $\mathbb{R}^3$. We are only concerned with feasibility, so the objective vector may be take to be an arbitrary (nonzero) vector. Any feasible is acceptable, even if the LP is unbounded. (In practice there are probably additional constraints placed on a, b, and c based on the technology of our cannon, but we will ignore this.) The LP returns feasible or unbounded, we report that it is possible to set the controls to satisfy the desired constraints. Otherwise (infeasible), we report that there is no solution. Thus, this problem can be solved in $O(n + m)$ expected time.

Again, if we want to be more formal (which is not required), we can express the inequalities as maximizing $f^{\mathsf{T}}x$ subject to $Ax \leq g$, where $f = (1, 0, 0)^{\mathsf{T}}$ (actually, any nonzero vector) and $Ax \leq g$ can be expressed as

$$\begin{bmatrix} -1 & -p_{1,x} & p_{1,x}^2 \\ \vdots & \vdots & \vdots \\ -1 & -p_{n,x} & p_{n,x}^2 \\ 1 & q_{1,x} & -q_{1,x}^2 \\ \vdots & \vdots & \vdots \\ 1 & q_{m,x} & -q_{m,x}^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \leq \begin{bmatrix} -p_{1,y} \\ \vdots \\ -p_{n,y} \\ q_{1,y} \\ \vdots \\ q_{m,y} \end{bmatrix}.$$

(c) First, we need to find an appropriate parameterization for the set of squares whose lower side is on the x-axis. We will represent a square as a pair (c, d) where (c, d) is the square's upper left corner. Because the square sits on the x-axis, the other corners are $(c, 0)$, $(c + d, 0)$ and $(c + d, d)$.)

In order for the square to lie entirely on or below the lower envelope of the halfplanes, it suffices that both of the upper corners of the square lie beneath all the lines. Thus, we have an LP in the variables $(c, d) \in \mathbb{R}^2$ with the $2n$ linear constraints:

$$d \;\leq\; a_i\, c + b_i \qquad \text{and} \qquad d \;\leq\; a_i(c + d) + b_i,$$

for $1 \leq i \leq n$. We can rewrite these in the more standard form as

$$-a_i\, c + d \;\leq\; b_i \qquad \text{and} \qquad -a_i c + (1 - a_i)d \;\leq\; b_i.$$

Since the square lies above the x-axis, we also have the constraint $d \geq 0$. (Alternatively, we could run the LP, maximizing d, and check at the end that $d \geq 0$.) The objective is the maximize the side length, that is, to maximize d. Thus, the objective vector is $(0, 1)$. Thus, we have an instance of LP in $\mathbb{R}^2$ with $2n$ constraints.

If this LP instance is infeasible, this means that the lower envelope of the lines lies entirely below the x-axis, and there is no such square. If the LP is unbounded, this means that we can have squares of arbitrarily large size. This can happen, for example, if all the line slopes are positive or all are negative. If the LP is feasible, and returns the point (c, d), we return the final square with corners $(c, 0)$, (c, d), $(c + d, 0)$ and $(c + d, d)$.

Again, if we want to be more formal (which is not required), we can express the inequalities as maximizing $f^{\mathsf{T}}x$ subject to $Ax \leq g$, where $f = (0,1)^{\mathsf{T}}$ and $Ax \leq g$ can be expressed as

$$\begin{bmatrix} -a_{1,x} & 1 \\ \vdots & \vdots \\ -a_{n,x} & 1 \\ -a_{1,x} & 1-a_1 \\ \vdots & \vdots \\ -a_{n,x} & 1-a_n \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \leq \begin{bmatrix} b_1 \\ \vdots \\ b_n \\ b_1 \\ \vdots \\ b_n \\ 0 \end{bmatrix}.$$

Solution 2:

(a) To prove that polar duality is order reversing, we observe that

$$\begin{array}{rcl} p = (c,d) \text{ lies} & \text{“inside” } \big| \text{ “on” } \big| \text{ “outside”} & \text{of line } \ell : ax + by = 1 \\ & \Longleftrightarrow & \\ ac + bd & \text{“}<\text{” } \big| \text{ “}=\text{” } \big| \text{ “}>\text{”} & 1 \\ & \Longleftrightarrow & \\ ca + db & \text{“}<\text{” } \big| \text{ “}=\text{” } \big| \text{ “}>\text{”} & 1 \\ & \Longleftrightarrow & \\ \ell^\circ = (a,b) \text{ lies} & \text{“inside” } \big| \text{ “on” } \big| \text{ “outside”} & p^\circ : cx + dy = 1. \end{array}$$

(b) Let $\ell_1 : a_1x + b_1y = 1$ and $\ell_2 : a_2x + b_2y = 1$. The point $p = (c,d)$ lies at their common intersection if and only if

$$a_1c + b_1d = 1 \qquad \text{and} \qquad a_2c + b_2d = 1$$

or equivalently,

$$ca_1 + db_1 = 1 \qquad \text{and} \qquad ca_2 + db_2 = 1$$

which is equivalent to saying that the points $\ell_1^\circ = (a_1, b_1)$ and $\ell_2^\circ = (a_2, b_2)$ lie on the line $p^\circ : cx + dy = 1$. Therefore, polar duality is incidence-preserving.

(c) Given a set of lines $L = \{\ell_1, \ldots, \ell_n\}$ let $K = \bigcap_{i=1}^n \ell_{i,\leq}$ denote the convex polygon defined by the intersection of their inner halfplanes (see Fig. 1(a)). Let $L^\circ = \{\ell_1^\circ, \ldots, \ell_n^\circ\}$ and define K° to be the convex hull of these dual points, that is, $K^\circ = \text{conv}(L^\circ)$ (see Fig. 1(b)).

To establish the relationship between K and K°, consider any point $p \in K$. The fact that p lies within K is equivalent to saying that p lies within all the inner halfplanes of L, that is $p \in \ell_{i,\leq}$, for all i. By (a), this is dually equivalent to saying that all the points of L° lie within p°'s inner halfplane, that is, $\ell_i^\circ \in p^\circ_{\leq}$, for all i. If a halfplane contains all the points of a set, then it contains their convex hull. We conclude that $p \in K$ if and only if $K^\circ \subseteq p^\circ_{\leq}$ (see Fig. 1(b)).

This implies that when a point $p = (a,b)$ travels along the boundary of K in a counterclockwise order (see Fig. 1(c)), its dual $p^\circ : ax + by = 1$ is a support line of K° that rotates

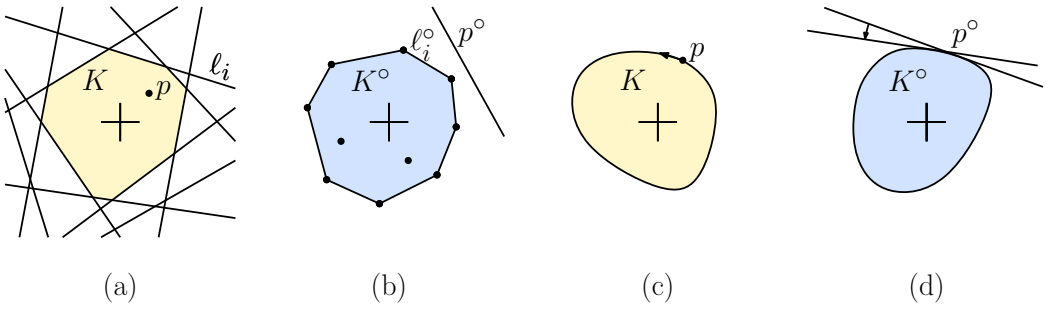


Figure 1: Support pair under the polar dual.

around the boundary of K° (see Fig. 1(d)). We assert that the support line also rotates in a counterclockwise direction. To see why, observe that as a point p travels around the boundary of a convex body in a counterclockwise direction, its associated support line rotates in counterclockwise manner. (This is a bit easier to visualize if you imagine that K is a smooth convex body, rather than a convex polygon.) It an easy exercise in geometry to see that the dual line $p^\circ : ax + by = 1$ is perpendicular to the vector $p = (a, b)$.[1] Thus, as a point $p = (a, b)$ travels in a counterclockwise direction around the boundary of K, the associated support line p° for K° rotates in a counterclockwise direction, which implies that its contact point with the boundary of K° also rotates in a counterclockwise direction (see Fig. 1(d)).

Solution 3:

(a) Let t denote the number of triangles in the triangulation. We will show that $t = 2v - h - 2$ and the number of edges is $e = 3v - h - 3$.

As observed in the hint, $3t = 2e - h$ (because $3t$ counts each internal edge twice and each edge of the h edges of the convex hull once). Thus, $e = (3t + h)/2$. By Euler's formula and the fact that $f = t + 1$, we have

$$\begin{aligned} 2 &= v - e + f = v - e + (t + 1) = v - \frac{3t + h}{2} + t + 1 \\ \Longrightarrow \quad t &= 2v - h - 2. \end{aligned}$$

Given that $f = t + 1 = 2v - h - 1$, by applying Euler's formula again, we have

$$\begin{aligned} 2 &= v - e + f = v - e + (2v - h - 1) \\ \Longrightarrow \quad e &= 3v - h - 3. \end{aligned}$$

(b) Let q denote the number of quadrangles in the quadrangulation. We assert that $q = v - \frac{h}{2} - 1$ and $e = 2v - \frac{h}{2} - 2$.

First, we observe that $4q = 2e - h$. This is because $4q$ counts every edge twice except for the h edges on the convex hull. Thus, $e = (4q + h)/2$. As before, $f = q + 1$. By Euler's formula

[1] This follows from the fact that the line passing through the vector has slope b/a and ℓ has slope $-a/b$. The slopes of perpendicular lines are negative reciprocals of each other.

and the fact that $f = q + 1$, we have

$$\begin{aligned} 2 &= v - e + f = v - e + (q+1) = v - \frac{4q+h}{2} + q + 1 \\ \implies q &= v - \frac{h}{2} - 1. \end{aligned}$$

Given that $f = q + 1 = v - (h/2)$, and again applying Euler's formula, we have

$$\begin{aligned} 2 &= v - e + f = v - e + v - \frac{h}{2} = 2v - e - \frac{h}{2} \\ \implies e &= 2v - \frac{h}{2} - 2. \end{aligned}$$

(c) We assert that the number of hull vertices in any quadrangulation must be odd. Since q and e are both positive integers, the answer to part (b) implies that $h/2$ is an integer, and hence the number of hull vertices h must be even.

Solution 4:

(a) Let d denote the height along the y-axis where the arrow is shot and let s denote the slope of the arrow's trajectory line. This defines the line equation $\ell : y = sx + d$. The upper/lower endpoints of the ith segment have the y-coordinates $c_{i,y} \pm \frac{h_i}{2}$. We will solve the problem of finding any solution vector $x = (d, s)^\mathsf{T} \in \mathbb{R}^2$ that satisfies the following constraints:

- The shot is made on the positive y-axis, so $d \geq 0$.
- The line passes above all the lower endpoints of all the segments: $sc_{i,x} + d \geq c_{i,y} - \frac{h_i}{2}$, for $1 \leq i \leq n$.
- The line passes below all the upper endpoints of all the segments: $sc_{i,x} + d \leq c_{i,y} + \frac{h_i}{2}$, for $1 \leq i \leq n$.

Since we seek any feasible answer, we do not care about the objective function. Let's say we minimize d, using the objective vector $(-1, 0)$. By the constraint $d \geq 0$, the LP cannot be unbounded. If the LP is feasible, we return pair d and directional vector $u = (1, s)$. Otherwise, the LP is infeasible, and we report that no such line exists. Since we have 2-dimensional LP with $O(n)$ constraints, the running time is $O(n)$ (in expectation).

(b) This can be solved by two invocations of LP in $\mathbb{R}^2$. The constraints will be exactly the same as in (a), but we need to be careful about the objective function. Observe that the line $y = sx + d$ crosses the ith segment at the y-value $p_{i,y} = sc_{i,x} + d$. Therefore, the average displacement of the line from all the centers is

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n}(p_{i,y} - c_{i,y}) &= \frac{1}{n}\sum_{i=1}^{n} p_{i,y} - \frac{1}{n}\sum_{i=1}^{n} c_{i,y} = \frac{1}{n}\sum_{i=1}^{n}(sc_{i,x} + d) - \frac{1}{n}\sum_{i=1}^{n} c_{i,y} \\ &= \left(\frac{1}{n}\sum_{i=1}^{n} c_{i,x}\right) s + d - \left(\frac{1}{n}\sum_{i=1}^{n} c_{i,y}\right) = As + d - B, \end{aligned}$$

where $A = \frac{1}{n}\sum_{i=1}^{n} c_{i,x}$ and $B = \frac{1}{n}\sum_{i=1}^{n} c_{i,y}$. Observe that A and B do not depends on either s or d and can be computed from the inputs. The result is a linear function in s and d. Unfortunately, we need to minimize its *absolute value*, and this is not a linear function. There are a couple of common techniques for handling absolute values in objective functions.

The first is that we make two calls to LP.

- First, we set the objective function to minimize $As + d$ and add the constraint $As + d \geq B$. (Formally, we are to maximize using the objective function given by the vector is $(-A, -1)^{\mathsf{T}}$.) Note that the LP cannot be unbounded. If it is feasible, we return this answer as the final result, since this is as close as we can get to B from above. If infeasible, we go to the second step.
- Second, we try again, but this time we maximize $As+d$ and add the constraint $As+d \leq B$ (that is, we maximize using the objective function given by the vector is $(A, 1)^{\mathsf{T}}$). As before, the LP cannot be unbounded. If it is feasible, we return this answer as the final result, since this is as close as we can get to B from below. Otherwise, we conclude that the entire problem is infeasible.

Since we have two instances of 2-dimensional LP with $O(n)$ constraints, the running time is $O(n)$ (in expectation).

The second approach is to employ a single instance of LP in $\mathbb{R}^3$ through the use of an additional *slack variable*. To do this, we create a new variable t which intuitively represents the "slack", that is, the maximum allowed deviation from the ideal value of B. We will run the LP in $\mathbb{R}^3$, with the variables (s, d, t). Our final value $As + d$ must lie within t units of B, that is,

$$As + d \;\leq\; B + t \quad \text{and} \quad As + d \;\geq\; B - t.$$

Because A and B are constants, these are both linear constraints in s, d, and t. (We think of t as being nonnegative, but we do not need to add the constraint $t \geq 0$, since this is implied by these two constraints.) Then to get the answer that is closest to B, we run a single LP, but with the objective of minimizing t. If infeasible, there is no shot that hits all the segments. The LP cannot be unbounded (since $t \geq 0$). If feasible, we output the solution implied by s and d.

Solution to the Challenge Problem:

(a) We will show that the expected number of points in the Pareto set of n uniformly distributed points in the plane is $\ln n + O(1)$. Let's assume that the points are in general position and that they have been sorted in *decreasing* order x-coordinates as $\langle q_1, \ldots, q_n \rangle$, where $q_i = (x_i, y_i)$. Let p_i denote the probability that q_i is in the Pareto set. Because x-coordinates are inversely sorted, q_i is in the Pareto set if and only if y_i is the maximum among $\{y_1, \ldots, y_i\}$. Because the points are uniformly distributed, the x- and y-coordinates of any point are statistically independent. Thus, the sequence of y-coordinates in the x-sorted point set is a random permutation of the y-coordinates. The probability that y_i is the maximum of the y-coordinates of $\{y_1, \ldots, y_i\}$ is clearly just $1/i$. Therefore, $p_i = 1/i$.

Next, observe that the expected number of points in the Pareto set is just

$$\sum_{i=1}^{n} p_i \;=\; \sum_{i=1}^{n} \frac{1}{i} \;=\; \ln n + O(1),$$

where the last equality follows from known bounds on the Harmonic series.

(b) We assert that the expected number of points on the convex hull of Q is at most 4 times the sizes of the individual Pareto sets. To see this, split the boundary of the convex hull of Q at its topmost, bottommost, leftmost, and rightmost points (see Fig. 2(a)). This splits the vertices of the convex hull into four parts, which can be called NW, NE, SW, and SE (for "northwest," "northeast", etc.). It is easy to see that that any vertex of the NE hull of Q is in the Pareto set of Q. (This follows from the fact that a supporting line through this vertex has a negative slope, which implies that none of the points of Q can lie in the upper-right quadrant with respect to this point (see Fig. 2(b)).)

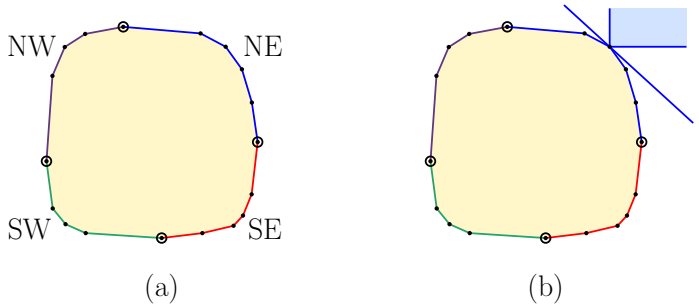


Figure 2: Pareto sets and convex hulls.

By symmetry, each of these sets has expected size at most $\ln n + O(1)$, so the total number of vertices is at most 4 times larger. Therefore, the expected number of vertices in the convex hull of Q is at most $4 \ln n + O(1) = O(\log n)$, as desired.