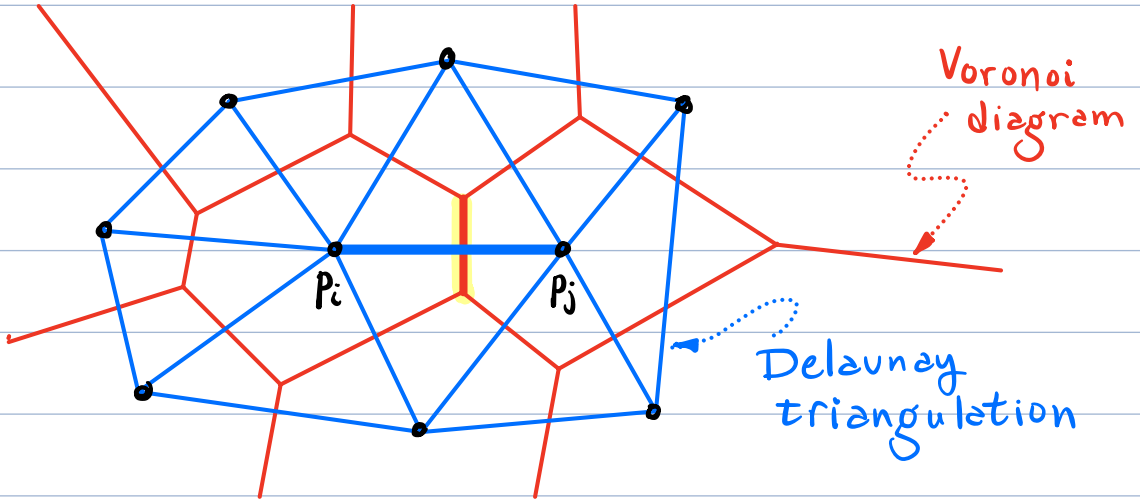
CMSC 754 - Computational Geometry
Lecture 11: Delaunay Triangulations I (Properties)

Previously we presented Voronoi diagrams. In this lecture, we consider its dual - Delaunay Triangulation



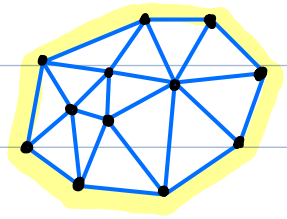
Delaunay Triangulation:

Given a set $P = \{p_1, \ldots, p_n\}$ of sites in $\mathbb{R}^2$, its Delaunay triangulation, $DT(P)$, is a cell complex whose vertices are the sites, and there is an edge $\overline{p_i p_j}$ iff the Voronoi cells $V(p_i)$ & $V(p_j)$ share a common edge. (i.e., $DT(P)$ & $Vor(P)$ are duals)

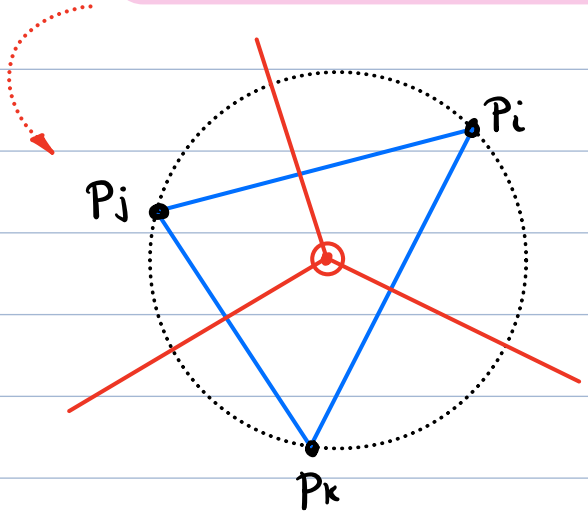
Properties:

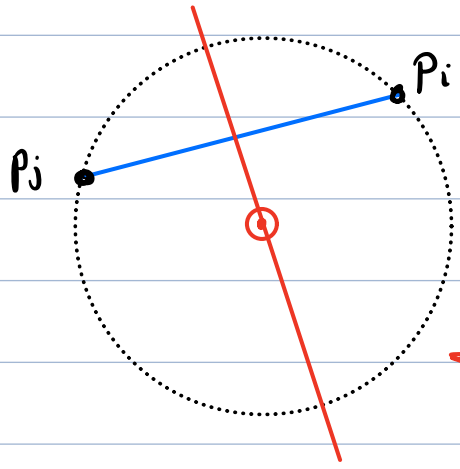
Triangulation: If the sites are in general position (no 4 cocircular) then internal faces are all triangles.

Hull: The boundary of the external face is the boundary of conv(P)



Circumcircle: The circumcircle of any triangle is empty (no sites in its interior)

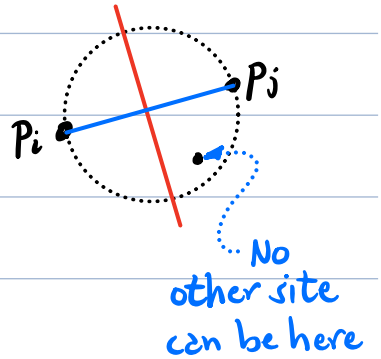




Empty circle: Sites p_i & p_j are adjacent iff there is an empty circle passing through p_i & p_j

Closest Pair: The closest pair of sites in P are Delaunay neighbors

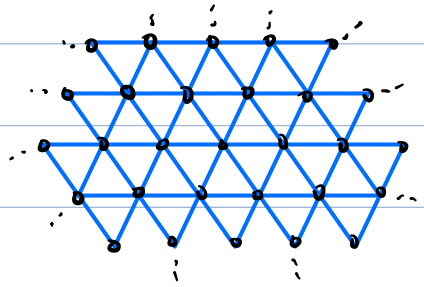
Why? If p_i & p_j are closest then circle with diameter $\overline{p_i p_j}$ must be empty.



Combinatorial Complexity:

There are at most $2n$ triangles and $3n$ edges

(Proof: Exercise in Euler's formula)



In $\mathbb{R}^d$ worst-case complexity is $O(n^{\lceil d/2 \rceil})$

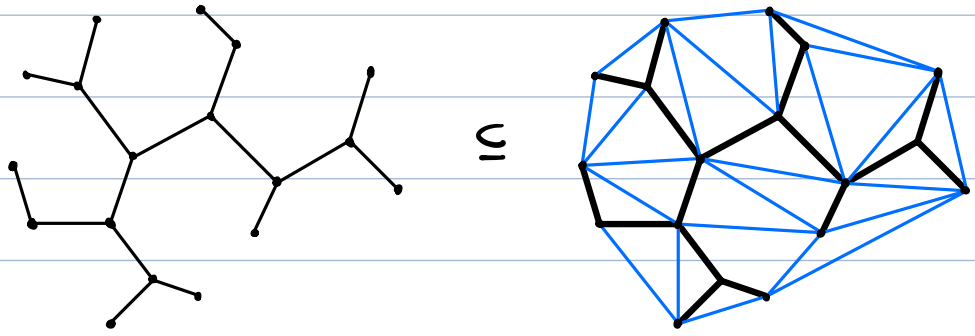
Euclidean Minimum Spanning Tree (EMST)

Def: Euclidean Graph - Given point set $P = \{p_1, \ldots, p_n\}$ this is the complete graph on P ($\binom{n}{2} = O(n^2)$ edges) with edge weights $w(p_i, p_j) = \|p_i - p_j\|$

Def: EMST(P) = minimum spanning tree of Euclid graph

Thm: EMST(P) is a subgraph of DT(P)

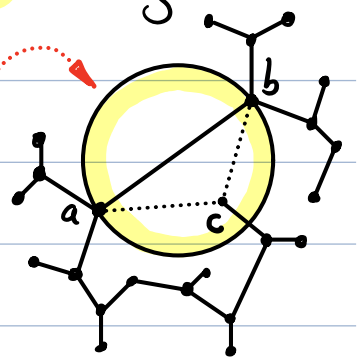
EMST(P) $\subseteq$ DT(P)



Proof: By contradiction. Suppose some edge $\overline{ab}$ of EMST(P) is not in DT(P).

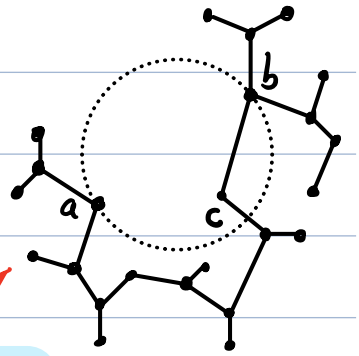
By the empty circle property, any circle through a & b is not empty

⇒ Consider circle with diameter $\overline{ab}$ & let c be any site inside this circle



Obs: $\|ac\| < \|ab\|$ & $\|bc\| < \|ab\|$

- Remove ab from EMST ⇒ 2 subtrees
- c is in one of these – WLOG assume in a's subtree
- Adding $\overline{cb}$ restores spanning tree.



$\text{cost}(\text{new MST}) = \text{cost}(\text{old MST}) - \|ab\| + \|bc\|$

$< \text{cost}(\text{old MST})$ [since $\|bc\| < \|ab\|$]

But this contradicts hypothesis that old MST is optimal.

□

Def: Minimum weight triangulation (MWT(P)): The triangulation that minimizes total edge length.

Is DT(P) = MWT(P)?

No! (Infamously claimed so, but there is a 4-point counterexample)

Shortest-Path Properties:

Def: Given a graph $G = (V, E)$, and $u, v \in V$, let

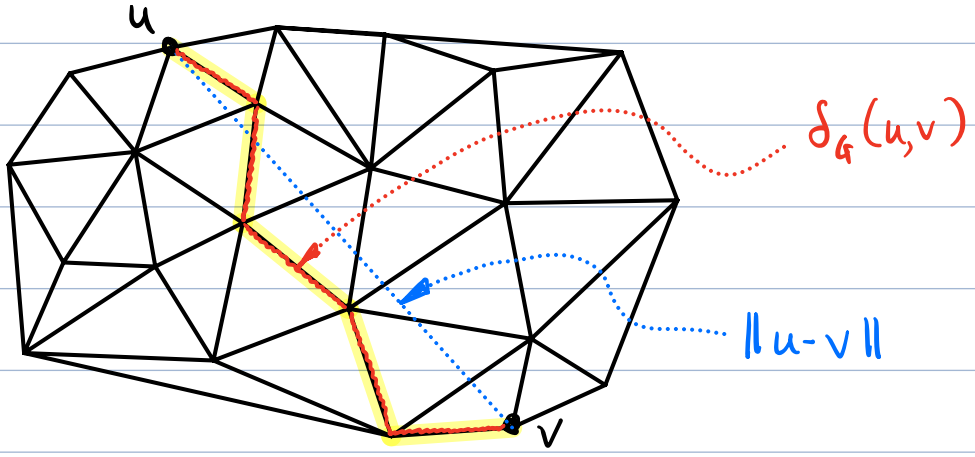
$\delta_G(u,v)$ = length of shortest path in G from u to v

Def: Given set $P \subseteq \mathbb{R}^2$ and $t \geq 1$, a Euclidean t-spanner is a straight-line graph G on P, s.t.

$$\delta_G(u,v) \leq t \cdot \|u - v\| \quad \forall u, v \in P$$

(called stretch factor)

Intuition: Path lengths are close to Euclid. dists



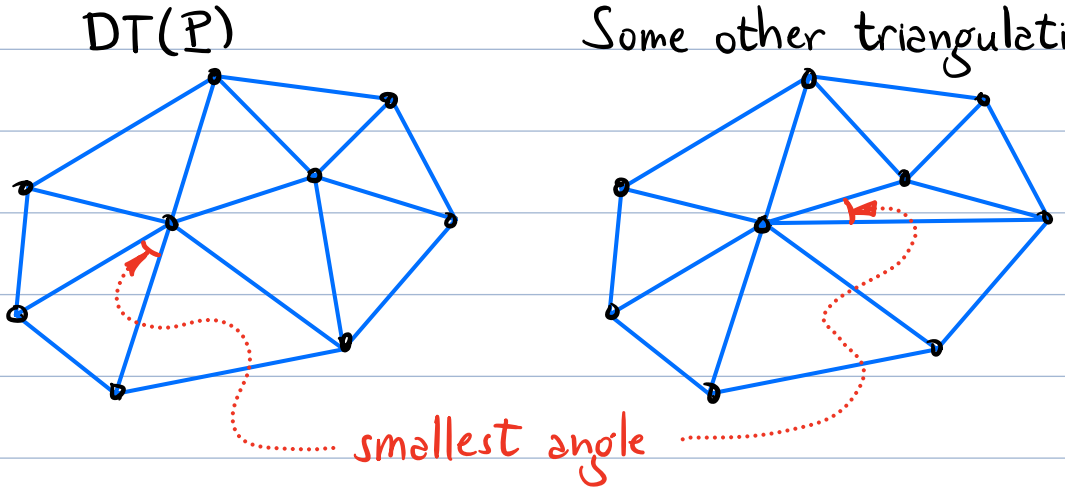
Thm (Xia 2013)

Given a set P of sites in $\mathbb{R}^2$, $DT(P)$ is a 1.998 Euclidean spanner:

$$\delta_{DT(P)}(u,v) \leq 1.998 \cdot \|u - v\| \quad \forall u, v \in P$$

High Quality Mesh Generation

- In numerical applications (e.g. finite element analysis + fluid dynamics) want to avoid very skinny triangles
- The Delaunay triangulation avoids small angles



Def: Given triangulation $T(P)$, let $ang(T(P))$ be the increasing sorted sequence of angles in $T(P)$

Theorem: For any point set P in $\mathbb{R}^2$ + any triangulation $T(P)$

$$ang(DT(P)) \geq_{LEX} ang(T(P))$$

where $\geq_{LEX}$ means lexicographical order

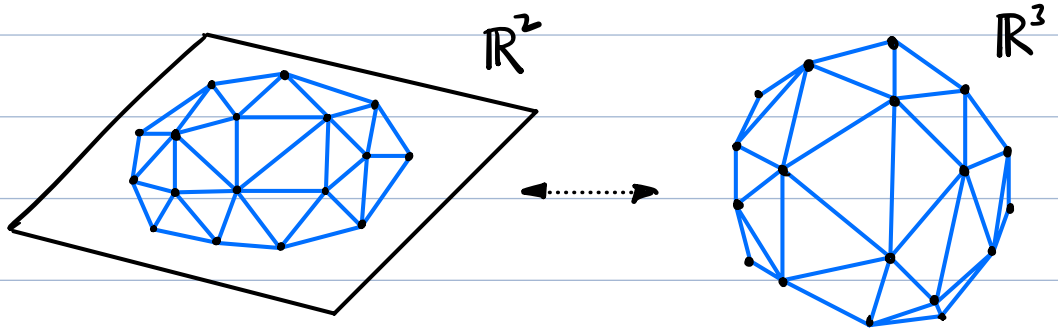
Thus, among all triangulations of P, $DT(P)$ maximizes the smallest angle, then second smallest, ...

(See lecture notes for proof)



Relationship to polytopes

Triangulations look much like polytopes - Coincidence?

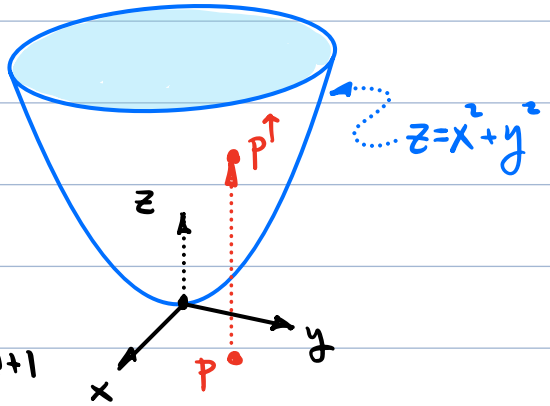


Lifting to the paraboloid:

Given $p \in \mathbb{R}^2$, define $p^\uparrow = (p_x, p_y, p_x^2 + p_y^2) \in \mathbb{R}^3$

Given $P \subseteq \mathbb{R}^2$, define

$P^\uparrow = \{ p^\uparrow \mid p \in P \} \subseteq \mathbb{R}^3$

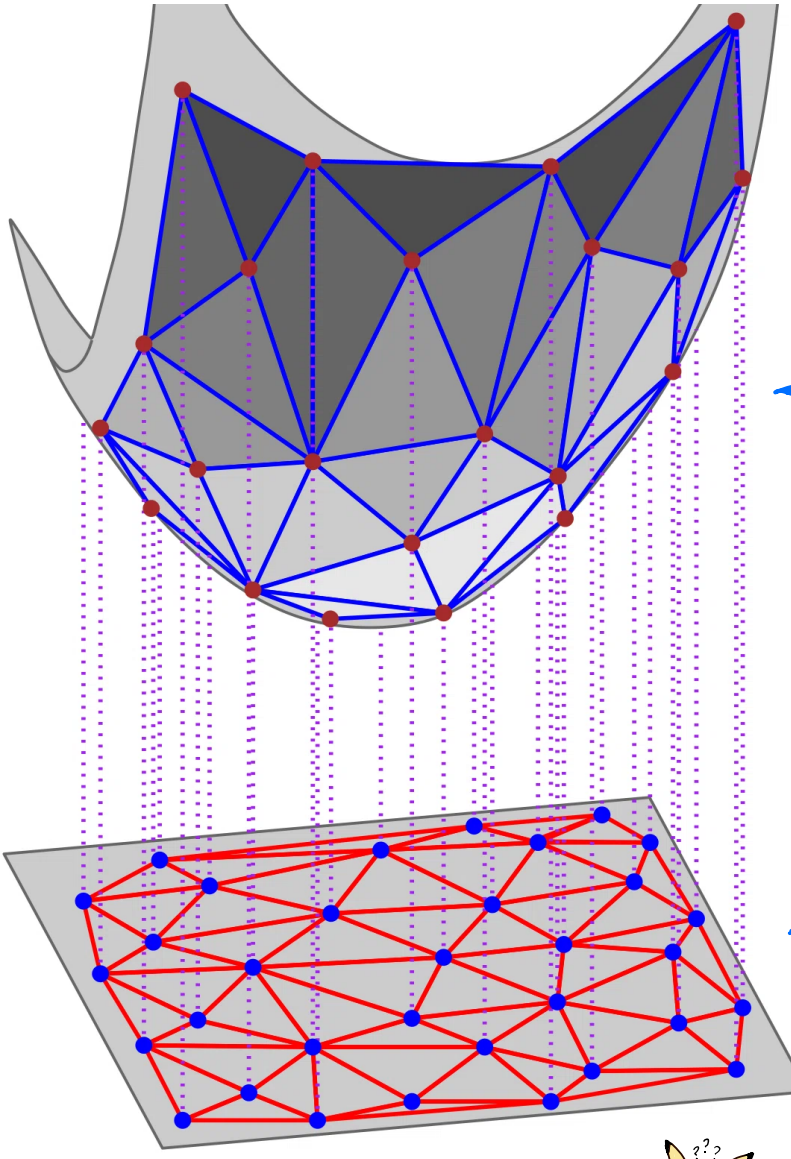


For $p \in \mathbb{R}^d$, define

$p^\uparrow = (p_1, \ldots, p_d, \sum p_i^2) \in \mathbb{R}^{d+1}$

Thm: Given a finite set $P \subseteq \mathbb{R}^d$, the faces of the lower hull of $P^\uparrow$ project vertically onto triangles of $DT(P)$.

Illustration-





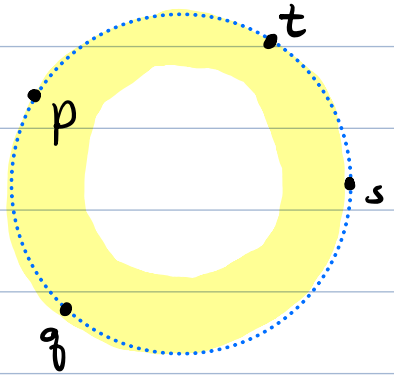
Why is this amazing?

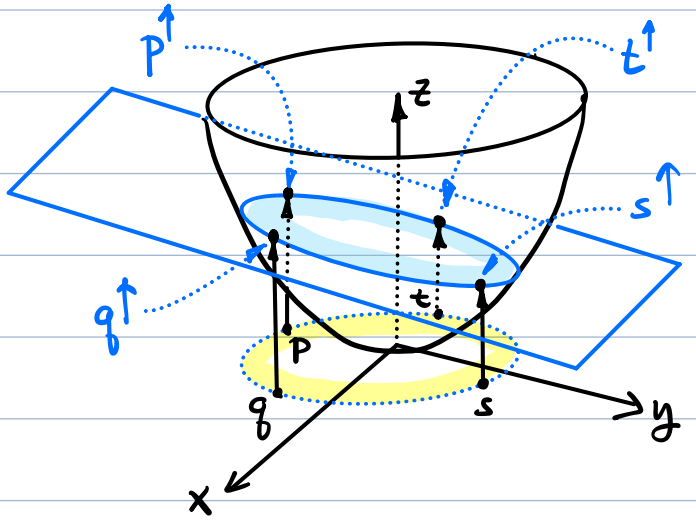
- Why should (curved) circles in $\mathbb{R}^2$ correspond to (flat) facets in $\mathbb{R}^3$?
- $DT(\mathbb{P})$ is invariant under translation but $\mathbb{P}^\uparrow$ is definitely not!

We'll prove only the main link:

cocircularity in $\mathbb{R}^2$ $\Leftrightarrow$ coplanarity in $\mathbb{R}^3$

Lemma: p, q, s, t are cocircular in $\mathbb{R}^2$ iff $p^\uparrow, q^\uparrow, s^\uparrow, t^\uparrow$ are coplanar in $\mathbb{R}^3$





Proof: We'll make use of the following fact from affine geometry:

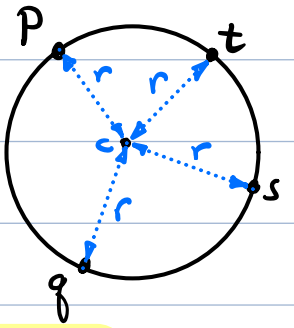
4 points p, q, s, t in $\mathbb{R}^3$ are coplanar iff

$$\det \begin{bmatrix} 1 & p_x & p_y & p_z \\ 1 & q_x & q_y & q_z \\ 1 & s_x & s_y & s_z \\ 1 & t_x & t_y & t_z \end{bmatrix} = 0$$

We'll show this holds for $p^\uparrow, q^\uparrow, s^\uparrow, t^\uparrow$

p, q, s, t are cocircular

p, q, s, t are cocicular iff $\exists$ center $c = (c_x, c_y)$ + radius r s.t.



$$r = \|p-c\| = \|q-c\| = \|s-c\| = \|t-c\|$$

$$\Leftrightarrow r^2 = \|p-c\|^2 = \|q-c\|^2 = \|s-c\|^2 = \|t-c\|^2$$

Let's just do p:

$$r^2 = (p_x - c_x)^2 + (p_y - c_y)^2$$

$$r^2 = (p_x^2 - 2c_x p_x + c_x^2) + (p_y^2 - 2c_y p_y + c_y^2)$$

$$0 = -2c_x p_x - 2c_y p_y + (p_x^2 + p_y^2) + c_x^2 + c_y^2 - r^2$$

$$\Leftrightarrow (1, p_x, p_y, p_x^2 + p_y^2) \cdot (c_x^2 + c_y^2 - r^2, -2c_x, -2c_y, 1) = 0$$

dot product

Applying to all 4 + using matrix notation:

$$\begin{bmatrix} 1 & p_x & p_y & p_x^2+p_y^2 \\ 1 & q_x & q_y & q_x^2+q_y^2 \\ 1 & s_x & s_y & s_x^2+s_y^2 \\ 1 & t_x & t_y & t_x^2+t_y^2 \end{bmatrix} \begin{bmatrix} c_x^2+c_y^2-r^2 \\ -2c_x \\ -2c_y \\ 1 \end{bmatrix} = 0$$

That is, there is a linear combination of $p^\uparrow, q^\uparrow, s^\uparrow, t^\uparrow$ in homogeneous coords that $= 0$

$\Leftrightarrow$ determinant of this matrix $= 0$

$\Leftrightarrow$ $p^\uparrow, q^\uparrow, s^\uparrow, t^\uparrow$ are coplanar (by prior fact) $\square$

Q: If DT(P) corresponds to lower hull of $P^{\uparrow}$ what triangulation corresponds to upper hull?

Hint: DT(P) is dual to the closest-point Vor. diagram What triangulation is dual to farthest-point Vor. diagram?

Summary:

Delaunay Triangulation + properties:

- Dual of the Voronoi diagram
- Empty circumcircle property
- EMST $\subseteq$ DT
  - But it is not the min. weight triangulation
- DT is a $\approx$ 1.998-Euclidean spanner
- DT maximizes the min angle (in $\mathbb{R}^2$)
  - But it does not minimize the max angle
- DT is vertical projection of lower hull of lifted points