Analysis of back-and-forth version of quicksort

Consider back-and-forth quicksort (described in hwk-6, prob 3) applied to n values. Let $T(n)$ be the average number of comparisons the algorithm uses. We know $T(0) = T(1) = 0$. The partition operation is done with exactly $n-1$ comparisons. Assume that the partition value turns out to have sorted rank (order position) q. Then we have its location, but we still have to recursively quicksort the $q-1$ smallest values and the $n-q$ largest values. So, for a fixed q, the remaining steps of quicksort take $T(q-1)+T(n-q)$ comparisons on average.

But the value of q is equally likely to be any integer from 1 to n. So, averaging over all possible values of q, the expected number of remaining steps of quicksort takes is $\sum_{q=1}^{n} \frac{1}{n}[T(q-1)+T(n-q)]$ comparisons.

We claim that $T(n) \leq an\ln n$ for some constant a and $n \geq 1$. Proof by constructive induction.

Base case: $n = 1$: $T(1) = 0$ and $a \cdot 1 \cdot \ln 1 = 0$.

Induction step: Assume it holds for all positive integers less than n. Then

$$
\begin{aligned}
T(n) &= \sum_{q=1}^{n} \frac{1}{n}[T(q-1)+T(n-q)] + n - 1 \\
&= \frac{1}{n}\sum_{q=1}^{n}[T(q-1)+T(n-q)] + n - 1 \\
&= \frac{1}{n}\sum_{q=1}^{n} T(q-1) + \frac{1}{n}\sum_{q=1}^{n} T(n-q) + n - 1 \\
&= \frac{1}{n}\sum_{q=0}^{n-1} T(q) + \frac{1}{n}\sum_{q=0}^{n-1} T(q) + n - 1 \\
&= \frac{2}{n}\sum_{q=0}^{n-1} T(q) + n - 1 \;=\; \frac{2}{n}\sum_{q=1}^{n-1} T(q) + n - 1 \\
&\leq \frac{2}{n}\sum_{q=1}^{n-1} aq\ln q + n - 1 \qquad \text{by IH} \\
&\leq \frac{2a}{n}\int_{1}^{n} q\ln q\,dq + n - 1 \qquad \text{by integral bound} \\
&= \frac{2a}{n}\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]\Big|_1^n + n - 1 \\
&= \frac{2a}{n}\left[\left(\frac{n^2}{2}\ln n - \frac{n^2}{4}\right) - \left(\frac{1^2}{2}\ln 1 - \frac{1^2}{4}\right)\right] + n - 1 \\
&= an\ln n - \frac{an}{2} + \frac{a}{2n} + n - 1 \\
&\leq an\ln n \qquad \textbf{if} \text{ the induction is to hold}
\end{aligned}
$$

Some straightfoward algebra now shows that $a \geq 2$, so $a = 2$ is the tightest upper bound we can get with this argument, and

$$T(n) \leq 2n \ln n \;=\; 2(\ln 2) n \lg n \;\approx\; 1.39 n \lg n$$