# GROWTH OF FUNCTIONS

We want to make more precise the notion that one function is insignificant with respect to another in the long run; that two functions are similar in the long run; etc.

We will consider two functions as being in the same family if one is a positive constant multiple of the other. Thus $x^2$ and $3.2 x^2$ are in the same family.

Now let f and g be functions; we in particular are concerned with the case where f and g are defined on positive integers, and we then write $f(n)$ and $g(n)$. We also are usually interested only in cases where the values of f and of g are positive.

We consider f to be dominated by g's family if one of the family members, say cg, can eventually keep up with f: there exists an $n_0$ and a $c>0$ such that $f(n) <= g(n)$ for all $n > n_0$. In effect, g tells f "I have a big sister who's at least as big as you, at least after a certain age". That is, $f(n)$ is less than or equal to some constant multiple of $g(n)$, from some point on.

The technical version is this:

$f = O(g)$ ["f is big-O of g'] means that there exists an $n_0$ and a $c > 0$ such that for all $n > n_0$, $f(n) <= cg(n)$.

EXAMPLE 1: $x = O(x^2)$

Here $x^2$ does not need to invoke help froma big sister, since $x^2$ is already bigger than x, at least for $x>1$. We just need t ofind constants c and $n_0$ to make this precise. But we let $c=1$ since we suspect $x^2$ itself will be big enough; and we let $n_0 = 1$ since then if $x>n_0$ we'll have $x>1$; and then $x^2 > x$ (by multiplying both sides by x).

EXAMPLE 2: $x^2 + 1 = O(x^2)$.

Note that although $x^2 + 1$ is always greater than $x^2$, if we consider a suitable multiple of $x^2$, say for instance $3x^2$, then this family member of $x^2$ outgrows $x^2 + 1$ for large x. Specifically:

$$x^2 + 1 \le 3(x^2) \quad \text{iff} \quad 1 \le 2x^2 \quad \text{iff} \quad x^2 \ge 1/2$$

iff

$$x \ge 1/\sqrt{2}$$

But $\sqrt{2} > 1$, so $1/\sqrt{2} < 1$, o for all $x > n_0$ we have $x > 1/\sqrt{2}$.

Then we choose $n_0 = 1$, and $c = 3$, and it follows that $x^2 + 1 = O(x^2)$.

EXAMPLE 3: $x^2 = O(x^2 - x)$.

We want $x^2 \le c[x^2 - x]$, so we hunt for c:

$$x^2 \le c[x^2 - x] \quad \text{iff} \quad 1 \le c[1 - 1/x]$$

iff

$$c \ge 1/[1 - 1/x] = 1/[(x-1)/x] = x/(x-1)$$

How can we guarantee that $c \ge x/(x-1)$? well, $x/(x-1)$ is always $> 1$, but also is $\le 2$ for large x. In fact, $\lim x/(x-1) = 1$ at infinity. At what point does $x/(x-1)$ become, and stay, less than 2? From calculus we find its derivative is $-1/(x-1)^2$ which is negative (for $x > 1$). Thus $x/(x-1)$ is decreasing (for $x > 1$). And at $x=2$, for example, $x/(x-1) = 2$, so it is less than 2 for $x > 2$. So let $c = 2$ and then we will have $c \ge x/(x-1)$ for $x > 2$. Thus $n_0$ can also be taken to be 2.

EXAMPLE 4: $x^3 \ne O(x^2)$

This example shows that not all functions are big-Oh of one another! We need to show that there is no big sister $c x^2$ that is bigger than $x^3$ from some point on. So, let $c x^2$ be any member of the family of $x^3$, and let $n_0$ be any positive real number. We'll find an $x > n_0$ such that $x^3 > c x^2$. We'll start with that last inequality and work backwards to see what x will work. We have

$$x^3 > c x^2$$

iff

$$1 > c/x \quad \text{(note that } x > 0 \text{ since } x > n_0\text{)}$$

iff

$$x > c$$

So all we have to do is choose $x > c$, and we are guaranteed that the family member $c x^2$ will not dominate $x^3$. But $c x^2$ was any family member at all; so $x^2$ has no family member that can keep up with $x^3$, and so $x^3$ is not big-Oh of $x^2$.

But it's worse than that (from $x^2$'s point of view): not only does it *not* have a family member that can keep up with $x^3$ -- and thus $x^3$ is not $O(x^2)$ -- but $x^3$ can keep up $x^2$ So, first of all, $x^2 = O(x^3)$ -- since we can let $c = 1$ and $x > n_0 = 1$ to guarantee that $x^2 < c x^3 = x^3$ . But $x^3$ is even faster-growing than that: it not only keeps up with $x^2$ but also with its whole family! There is a special notation fof this: $x^2 = o(x^3)$, or "$x^2$ is little-oh of $x^3$".

Definition: $f = o(g)$ means that for every $c>0$ there is an $n_0$ such that for all $x > n_0$, $f(x) < c g(x)$.

That is, every member of g's family beats f, even when $c < 1$. Note that this is the same as saying that g beats every member of f's family, since we can just divide by c to get $1/c f(x) < g(x)$, where $1/c f(x)$ is simply any member of f's family.

So $f = o(g)$ is a very strong statement, stronger than $f = O(g)$. The latter says some multiple g can keep up with f; the former says every multiple of g can do so, even tiny multiples. To do that, g must indeed grow very fast compared to f, such as $x^3$ does compared to $x^2$.

Big-Oh and little-oh are the essential notions of growth of functions, but there are a few others notations that are sometimes useful, which ae easily define in terms of these two:

$f = Theta(g)$ means $f = O(g)$ and also $g = O(f)$

$f = Omega(g)$ means $g = O(f)$

$f = omega(g)$ means $g = o(f)$

EXAMPLES

$x^2 = Theta(x^2 - x)$

$x = o(x^2)$

$x^2 = omega(x)$

Theta can also be defined as follows:

f = Theta(g) iff there are constants c_1>0, c_2>0, and n_0, such that for all n > n_0,  c_1 g(n) <= f(n) <= c_2 g(n) .  Note that c_1 and c_2 give the two Big-Oh halves of our original definition of Theta.

Now for a more practical example: runtime of insertion sort.  Recall that

 T_INSRT (n) <= 0.5 n^2 - 0.5 n (upper bound, worst case)

and

 n-1 <= T_INSRT (n) (lower bound, best case)

So,

 T_INSRT (n) = O(n^2)

and also

 n = O(T_INSRT (n))

The latter can be more nicely expressed as T_INSRT (n) = Omega(n).

Now let's give an overview of common functions and how they "line up" in terms of growth.  The following list is in order of slowest-growing to largest, where f = o(g) for each side-by-side pair f,g:

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   1/n   1   lg n   n   n*lg n   n^2   ...   n^k   ...   2^n   n!   n^n
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In many cases, one can prove the little-oh relation by the following means:  show lim f/g = 0 at infinity.  The reason this can be used is that since we want to show f < cg for all c, this means we'll be showing f/g < c, for all c, ie f/g is smaller than any positive c, eventually (ie for x > some n_0).  But this is just the definition of a limit = 0 at infinity.

EXAMPLE:  lg n = o(n).  For consider lim (lg n)/n at infinity. By L'Hopital's Rule, this is lim (1/n)/1 = lim 1/n = 0.

This also works the other way around: if the lim is not 0, then little-oh fails; for instance, lim (n - lg n)/n = lim [1 - (lg n)/n]

= 1, so n - lg n is *not* o(n).

EXAMPLE: lg(lg n) = o (lg n). Here lim lg(lg n) / lg n =
lim 1/(lg n) 1/n / 1/n = lim 1/(lg n) = 0 since lim lg n = infinity.

In cases involving n!, there is a useful formula -- Stirling's formula:

n! = sqrt(2 pi n) (n/e)^n (1 + Theta(1/n))

What this means is that, for large n, we have the approximation

n! ~ sqrt(2 pi n) (n/e)^n (1 + f(n))

where f(n) = Theta(n), ie, f(n) is a function that, for large n,
behaves like 1/n, and hence tends to zero.

One can also sometimes use limits to show a Theta relation, but one
must be careful since it does not always apply.

EXAMPLE: 2n^2 = Theta(n^2 -n). Note that lim [2n^2 / (n^2 - n)] at
infinity is also 1 / lim [(n^2 - n) / 2n^2] which in turn is
1 / lim [1/2 - 1/2n] = 1 / (1/2) = 2. So the two functions, 2n^2 and
n^2 - n, stay within a factor of about 2 of each other, for large n.
In particular, for large n, each is less than 3 times the other, and
this shows each is Big-Oh of the other, hence the Theta relation
holds.

In general whenever a finite limit of the ratio of two functions
exists at infinity, they are Theta of each other; but the converse
need not hold.