CMSC250, Spring 2004 Homework 12 Answers

Due Wednesday, April 28 at the beginning of your discussion section.

You must write the solutions to the problems single-sided on your own lined paper, with all sheets stapled together, and with all answers written in sequential order or you will lose points.

1. Prove or disprove the following:

   (a) $f : (0, \infty) \to \mathbf{R}$ defined by $f(x) = \ln(x)$ is 1-1.

   Answer: TRUE. Proof:
   Let x_1 and x_2 be arbitrary numbers in $(0, \infty)$ such that $f(x_1) = f(x_2)$.
   By the definition of f, $f(x_1) = \ln(x_1)$ and $f(x_2) = \ln(x_2)$.
   By substitution, $\ln(x_1) = \ln(x_2)$.
   By taking antilogs, $e^{\ln(x_1)} = e^{\ln(x_2)}$.
   By simplifying, $x_1 = x_2$.
   By closing the conditional world and generalizing from the generic particular, $\forall x_1, x_2 \in (0, \infty)\ [f(x_1) = f(x_2)] \to [x_1 = x_2]$.
   Therefore, f is 1-1 by the definition of 1-1.

   (b) $f : (0, \infty) \to \mathbf{R}$ defined by $f(x) = \ln(x)$ is onto.

   Answer: TRUE. Proof:
   Let y be arbitrary in $\mathbf{R}$.
   Define x to be e^y.
   Because $e > 0$, we know that $x = e^y$ is greater than zero, and therefore is in $(0, \infty)$.
   By the definition of f, $f(x) = f(e^y) = \ln(e^y) = y \cdot \ln(e) = y$ by substitution, algebra, and properties of logarithms.
   By generalizing from the generic particular, $\forall y \in \mathbf{R}\ \exists x \in (0, \infty)\ f(x) = y$.
   Therefore, f is onto by the definition of onto.

2. Let A and B be sets, and $f : A \to B$ be the function $f(x) = x^2$. For each of the four following questions, define the sets A and B such that f is

   (a) 1-1 and onto.

   Answer: Let $A = B = \mathbf{R}^{\geq 0}$. Others are possible, too.

   (b) 1-1 but not onto.

   Answer: Let $A = \mathbf{R}^{\geq 0}$ and $B = \mathbf{R}$. Others are possible, too.

   (c) onto but not 1-1.

   Answer: Let $A = \mathbf{R}$ and $B = \mathbf{R}^{\geq 0}$. Others are possible, too.

   (d) neither 1-1 nor onto.

   Answer: Let $A = B = \mathbf{R}$. Others are possible, too.

3. Let f be an arbitrary bijective function from $\mathbf{R}$ to $\mathbf{R}$.

   (a) Define a function $g_1 : \mathbf{R} \to \mathbf{R}$ in terms of f so that g_1 is 1-1 but not onto. Prove that g_1 satisfies this criteria.

   Answer: There are many answers. Here's one.
   Let $g_1(x) = e^{f(x)}$.

   First, we prove that g_1 is 1-1:
   Let a and b be arbitrary real numbers and assume $g_1(a) = g_1(b)$.
   By definition of g_1, $g_1(a) = e^f(a)$ and $g_1(b) = e^f(b)$.
   By substitution, $e^f(a) = e^f(b)$.
   By taking the natural log of both sides, we get $f(a) = f(b)$.
   Since f is 1-1, we know $a = b$. By closing the conditional world and generalizing from the generic particular,
   $\forall a, b \in \mathbf{R}\ [g_1(a) = g_1(b)] \to [a = b]$.
   Then g_1 is 1-1 by the definition of 1-1.

   Now we show g_1 is not onto:
   Let $y = 0$. Zero is a real number, but since $e > 0$, raising e to any power will never give you zero. Therefore, g_1 is not onto.

   (b) Define a function $g_2 : \mathbf{R} \to \mathbf{R}$ in terms of f so that g_2 is onto but not 1-1. Prove that g_2 satisfies this criteria.

   Answer: Again, there are many answers. Here's one.
   Let $g_2(x) = f(\ln|x|)$ if $x \neq 0$ and 0 otherwise.

   First, we prove that g_2 is onto:
   Let y be arbitrary in $\mathbf{R}$.
   Define x to be $e^{f^{-1}(y)}$.
   Then $g_2(x) = g_2(e^{f^{-1}(y)}) = f(\ln|e^{f^{-1}(y)}|) = f(f^{-1}(y)) = y$.
   We can do this because f^{-1} exists since f is a bijection, and since $e > 0$, $e^{f^{-1}(y)}$ will never be zero.
   By generalizing from the generic particular, $\forall y \in \mathbf{R}\ \exists x \in \mathbf{R}\ g_2(x) = y$.
   Therefore, g_2 is onto by the definition of onto.

   Now we show g_2 is not 1-1:
   Let $x_1 = -1$ and $x_2 = 1$.
   Then $f(x_1) = \ln|-1| = \ln|1| = f(x_2)$, but obviously $x_1 \neq x_2$.

4. Let D denote the set of odd integers.
   That is, $D = \{n \in \mathbf{Z} \mid n = 2k + 1 \text{ for some integer } k\}$.

   (a) Define a bijective function $h_1 : D \to \mathbf{Z}$. Prove that h_1 is a bijection.

   Answer: Let $h_1(x) = (x - 1)/2$.

   First we prove h_1 is 1-1:
   Let a and b be arbitrary odd numbers and assume $h_1(a) = h_1(b)$.
   By definition of h_1, $h_1(a) = (a - 1)/2$ and $h_1(b) = (b - 1)/2$.

By substitution, $(a-1)/2 = (b-1)/2$.
By multiplying both sides by 2 and adding 1, $a = b$.
By closing the conditional world and generalizing from the generic particular, $\forall a, b \in D\ [h_1(a) = h_1(b)] \rightarrow [a = b]$.
Then h_1 is 1-1 by the definition of 1-1.

Now we show h_1 is onto:
Let y be arbitrary in $\mathbf{Z}$.
Define x to be $2y+1$. We know $2y+1 \in D$ by the definition of odd.
Then $h_1(x) = ((2y+1)-1)/2 = (2y)/2 = y$.
By generalizing from the generic particular, $\forall y \in \mathbf{Z}\ \exists x \in D\ h_1(x) = y$.
Therefore, h_1 is onto by the definition of onto.

(b) Define a bijective function $h_2 : \mathbf{Z} \rightarrow D$. Prove that h_2 is a bijection.

Answer: Let $h_2(x) = 2x + 1$.

First we prove h_1 is 1-1:
Let a and b be arbitrary integers and assume $h_2(a) = h_2(b)$.
By definition of h_2, $h_2(a) = 2a + 1$ and $h_2(b) = 2b + 1$.
By substitution, $2a + 1 = 2b + 1$.
By subtracting 1 from both sides and dividing by 2, $a = b$.
By closing the conditional world and generalizing from the generic particular, $\forall a, b \in \mathbf{Z}\ [h_2(a) = h_2(b)] \rightarrow [a = b]$.
Then h_2 is 1-1 by the definition of 1-1.

Now we show h_2 is onto:
Let y be arbitrary in D.
Define x to be $(y-1)/2$.
Since y is odd, there exists some integer k such that $y = 2k + 1$. Therefore, $x = (y-1)/2 = ((2k+1)-1)/2 = k$, which means x is an integer.
Then $h_2(x) = h_2((y-1)/2) = 2((y-1)/2) + 1 = y - 1 + 1 = y$.
By generalizing from the generic particular, $\forall y \in D\ \exists x \in \mathbf{Z}\ h_2(x) = y$.
Therefore, h_2 is onto by the definition of onto.