CMSC250, Spring 2004 Homework 6 Answers

Due Wednesday, March 10 at the beginning of your discussion section.

You must write the solutions to the problems single-sided on your own lined paper, with all sheets stapled together, and with all answers written in sequential order or you will lose points.

1. Let $p_1, p_2, \ldots, p_n$ be distinct prime numbers such that $p_1 = 2$ and $n > 1$, and let $a = p_1 p_2 \cdots p_n$. Prove that $a \equiv_4 2$.

   Hint: Use the quotient-remainder theorem, and eliminate the other possibilities.

   Answer: Proof:
   Let $p_1, p_2, \ldots, p_n$ be distinct prime numbers such that $p_1 = 2$ and $n > 1$, and let $a = p_1 p_2 \cdots p_n$. By the quotient-remainder theorem, we know there exist unique integers q and k such that $a = 4q + k$, $0 \leq k < 4$.
   Then $a = 4q \vee a = 4q + 1 \vee a = 4q + 2 \vee a = 4q + 3$ by enumerating the possibilities.

   Case 1: $a = 4q$.
   We know that $a = 2p_2 \cdots p_n$ by substitution.
   We also know that $a = 4q = 2^2 q$ by algebra.
   Let's set these two representations of a equal to each other: $2p_2 \cdots p_n = 2^2 q$.
   Let x be the number of factors of 2 in the standard factored form of q.
   So $q = 2^x q'$ for some nonnegative integer x and some integer q', and $2 \nmid q'$.
   So now we have this equality: $2p_2 \cdots p_n = 2^{2+x} q'$.
   Since none of the $p_2, \ldots, p_n$ are two (because $p_1 = 2$ and all of them are distinct), there is only one factor of two on the left side. The right side has at least two factors of 2, possibly more. Therefore, by the unique factorization theorem, $1 = 2 + x$.
   By algebra, $x = -1$, which is a contradiction since we stated x is a nonnegative integer.

   Case 2: $a = 4q + 1$ or $q = 4q + 3$. (We're combining these two cases because they are so similar.)
   Since 1 and 3 are both odd, we can write a as $4q + (2k + 1)$ for some integer k.
   $a = 2(2q + k) + 1$ by algebra.
   Since $2q + k \in \mathbf{Z}$ by closure of the integers under multiplication and addition, a is odd by the definition of odd.
   However, since $a = 2p_2 \cdots p_n$ and $p_2 \cdots p_n \in \mathbf{Z}$ by closure of the integers under multiplication, a is even by the definition of even.
   This is a contradiction since no integer can be both even and odd.

   So $a \neq 4q$ and $a \neq 4q + 1$ and $a \neq 4q + 3$. By applying conjunctive simplification and disjunctive syllogism three times, we learn that $a = 4q + 2$.
   Therefore, $a - 2 = 4q$ by algebra, which means $a \equiv_4 2$ by the definition of mod.

2. An integer is called *squarefree* if it is not divisible by the square of any prime number. What can you deduce about the standard factored form of a squarefree number?

   Hint: Examine the exponents.

   Answer: All the exponents in the standard factored form must be either zero or one.

3. Prove that any integer $a > 1$ can be written as a either perfect square, a squarefree integer, or the product of a squarefree integer and a perfect square.

   Answer: Proof:
   Let a be an integer greater than 1.
   Assume that a cannot be written as either a perfect square nor a squarefree integer.
   Since a is an integer, it can be factored over the primes. We will split the prime factors into those with even exponents and those with odd exponents.
   $a = (p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n})\,(q_1^{d_1} q_2^{d_2} \cdots q_m^{d_m})$ where $p_i, q_i \in \mathbf{Z}^{\text{prime}}$, $e_i, d_i \in \mathbf{Z}^{\text{nonneg}}$, $m \in \mathbf{Z}^+$, $n \in \mathbf{Z}^{\text{nonneg}}$, and all the e_i's are even and the d_j's are odd.
   We can do this for the following reasons:

   - Since a is not a perfect square, there must be at least one odd exponent in the standard factored form.
   - Since a is not squarefree, there must be some exponent that is greater than one.

   Since all of the e_i's are even, each one can be written as $2x_i$ for some integer x_i.
   Since all of the d_j's are odd, each one can be written as $2y_j + 1$ for some integer y_j.
   So $a = (p_1^{2x_1} p_2^{2x_2} \cdots p_n^{2x_n})(q_1^{2y_1+1} q_2^{2y_2+1} \cdots q_m^{2y_m+1})$ by substitution, and this equals
   $\left[(p_1^{2x_1} p_2^{2x_2} \cdots p_n^{2x_n})(q_1^{2y_1} q_2^{2y_2} \cdots q_m^{2y_m})\right]\left[q_1^1 q_2^1 \cdots q_m^1\right]$ by algebra, and this equals
   $[(p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n})(q_1^{y_1} q_2^{y_2} \cdots q_m^{y_m})]^2 \left[q_1^1 q_2^1 \cdots q_m^1\right]$ by algebra.
   The first factor is clearly a perfect square, and the second term is a squarefree integer because all of the exponents in the standard factored form are 1.
   So a can be written as as the product of a squarefree integer and a perfect square.
   By closing the conditional world, if a is neither a perfect square nor squarefree, then it is the product of a squarefree integer and a perfect square.
   By applying the definition of implication, a is either a perfect square, a squarefree integer, or the product of a squarefree integer and a perfect square.
   By generalizing from the generic particular, any integer greater than 1 can be written as as the product of a squarefree integer and a perfect square.

4. Prove that for all real numbers x, $x - 1 < \lfloor x \rfloor \le x$.

   Answer: Proof:
   Let x be a real number.
   By the definition of floor, $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$.
   Since this is an "and" statement, we can use conjunctive simplification and get
   $\lfloor x \rfloor \le x$.
   If we subtract one from both sides of $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$, we get
   $\lfloor x \rfloor - 1 \le x - 1 < \lfloor x \rfloor$.
   Since this is an "and" statement, we can use conjunctive simplification and get
   $x - 1 < \lfloor x \rfloor$.
   Since $x - 1 < \lfloor x \rfloor$ and $\lfloor x \rfloor \le x$, by the transitivity of "$<$", we can combine these two inequalities to get
   $x - 1 < \lfloor x \rfloor \le x$.

5. Write the first four terms of each of the following sequences:

   (a) $\forall j > 1,\ s_j = j(j-1)$
   Answer: 2, 6, 12, 20.

(b) $\forall n \geq 0,\ r_n = \dfrac{n}{n!}$

Answer: 0, 1, 1, 1/2.

6. Reduce each of the following expressions to a single numeric value:

(a) $\displaystyle\sum_{i=1}^{4} \frac{-1}{i^2+i} = -4/5.$

(b) $\displaystyle\prod_{h=0}^{10} \frac{h}{h!+h+1} = 0.$

(c) $\displaystyle\sum_{m=1}^{3} \left(\prod_{n=1}^{m} \frac{m}{n} \right) = 15/2.$

7. Write each of the following using sum and/or product notation:

(a) $n + 0 - n - 1 + n + 2 - n - 3 + n + 4$

Answer: $\displaystyle\sum_{i=0}^{4} \big((-1)^i \cdot (n+i) \big)$

(b) $\sqrt{5}(1) + 2(1+2) + \sqrt{3}(1+2+3) + \sqrt{2}(1+2+3+4) + 1(1+2+3+4+5)$

Answer: $\displaystyle\sum_{i=1}^{5} \left(\sqrt{6-i} \cdot \sum_{j=1}^{i} j \right)$